AD-A077 577 OHIO STATE UNIV RESEARCH FOUNDATION COLUMBUS F/G 20/11
MULTIPLE SCATTERING OF ACOUSTIC, ELECTROMAGNETIC AND ELASTIC WA--ETC(U)
SEP 79 V K VARADAN N00014-78-C-0559
UNCLASSIFIED OSURF-761261/711400 NL

1 OF 1
ADA
077 577

END
DATE
FILMED
1 -80
DDC

NR 064-607
code 474

RF Project 761261/711400
and 761251/711374
(9) Technical Report

LEVEL II

AD A077577

the
ohio
state
university

research foundation
1314 kinnear road
columbus, ohio
43212

DDC
RECEIVED
DEC 3 1979
E

(14) OSURF-761261/711400,
OSURF-761251/711374

(6) MULTIPLE SCATTERING OF ACOUSTIC, ELECTROMAGNETIC
AND ELASTIC WAVES.

(10) Vijay K./Varadan
Department of Engineering Mechanics

(12) 33

U. S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
Boulder, Colorado 80302

Grant No. 04-78-B01-21

DEPARTMENT OF THE NAVY
Office of Naval Research
Arlington, Virginia 22217

(15) Contract No. N00014-78-C-0559,
NOAA-04-78-B01-21

DDC FILE COPY

(11) September 1979

267 360   79 10 09 139

MULTIPLE SCATTERING OF ACOUSTIC, ELECTROMAGNETIC AND ELASTIC WAVES

V.K. Varadan
Wave Propagation Group
Department of Engineering Mechanics and Atmospheric Sciences Program
The Ohio State University, Columbus, Ohio 43210

ABSTRACT

In this article we present a multiple scattering analysis of the coherent wave propagation through an inhomegeneous medium consisting of either random or periodic distribution of scatterers of arbitrary shape. Both specific and random orientations of the scatterers are considered. The mathematical unity inherently present in the T-matrix formalism for the three wave fields, namely acoustic, electromagnetic and elastic, is employed in conjunction with suitable averaging procedures to formulate a self-consistent multiple scattering theory. For a random distribution of scatterers we use a configurational averaging procedure, while for a periodic distribution, we use a suitable lattice sum based on crystallographic theory. The information about the orientation of the scatterers has been incorporated into the T-matrix of the scatterer itself thus making formalism a convenient computational scheme to study the anisotropic effects in an inhomegeneous medium. The statistically averaged equations obtained by the analysis are then solved by using Lax's quasicrystalline approximation to obtain the bulk or effective properties of the medium. Numerical results are presented for propagation speeds, attenuation and frequency dependent elastic properties for a range of frequencies to demonstrate the broad applicability of the T-matrix method.

I. INTRODUCTION

The subject of wave propagation and scattering in inhomogeneous media has become increasingly important in many fields of engineering and science, for e.g., in studies of bubbles in a fluid, distribution of flaws in solids (NDE), ionospheric irregularities, geophysical exploration, artificial dielectrics, millimeter wave propagation in ocean fogs and mists, and through rain, cloud and hail particles, underwater signal transmission, porous media, composites, attenuation of waves in biological media, remote sensing, etc. All of these problems are characterized by a suitable statistical description of the media. The similarity in statistical descriptions and the mathematical unity present in the description of all three classical wave fields, namely acoustic, electromagnetic and elastic, may be taken into consideration in formulating a unifying approach to all these problems. In this article, we present one such unifying approach based on the T-matrix or the null field method that has been recently formulated for all three fields and

promises to be an efficient computational scheme to analyze the scattering of waves from several different geometries.

At any point in a random medium, the total wave fields can be considered as a sum of two components, viz, a coherent or average wave (which is the statistical average over all possible configurations of the scatterers with regard to location and state) and an incoherent component due to changes in scatterer positions and states from configuration to configuration. The averages of the square of magnitude of the coherent and incoherent fields are called the coherent and incoherent intensities, respectively. The total intensity is equal to the sum of coherent intensity and incoherent intensity. For a plane wave incident on a medium containing random particles, the coherent intensity attenuates due to scattering and absorption. Incoherent scattering effects introduce 'noise' into the system and cause fluctuations in the coherent received amplitude and phase. In radar meteorological applications it is important to assess the incoherent scattered intensity relative to the total intensity in order to relate theoretical and experimental results (e.g., power law relations between attenuation and rainfall rate). Propagation of the coherent wave is generally expressed in terms of a bulk propagation coefficient characterizing the scatterer filled medium. Incoherent effects are usually determined by solving 'approximate' integral equations or by solving special forms of the radiative transfer equations. Such formulations are generally valid under conditions of sparse concentrations and/or weak multiple scattering for either Rayleigh scatterers or large scatterers which scatter primarily in the forward direction. Incoherent scattering is beyond the scope of this present paper, and those who are interested in such analysis may refer to Ishimaru (1). Here, our attention focuses on formulating a multiple scattering theory for studying coherent wave propagation through an inhomogeneous medium consisting of either random or periodic distribution of scatterers of arbitrary shape.

Several theories and models on diffraction and scattering of waves have been pursued ever since Rayleigh's (2) analytical treatment of scattering of waves by randomly distributed particle to study the color of the sky. We cite here papers that are related to our present analysis (3-21). The limitations, difficulties and advantages of various approaches are discussed in a recent review article (3). We also refer to (1) and the references cited therein. Since scattering theory starts with discrete ensemble of inhomogeneities before statistical averaging is carried out, the specific geometry and orientation of each scatterer can be easily incorporated into the formulation. This is not possible with continuum theories. The additional advantage of the scattering theory approach is that it is the counterpart of actual experiments performed for coherent wave propagation in heterogeneous media using elastic, electromagnetic or acoustic waves as appropriate.

In multiple scattering theories, approximations are usually made for the geometry and size of the scatterer relative to the wavelength of incident waves, and the distribution of scatterers in the medium. The approximations with respect to geometry and size of the scatterers are related. If the scatterers are small compared to the incident wavelength, one usually obtains the gross scattering properties of the medium. This is the so-called Rayleigh or low frequency limit, and yields corrections to point scatterers. As far as the distribution of the scatterers is concerned, we either have regular arrays of scatterers or a random distribution. In the first case, we employ suitable lattice sum while in the latter case, we use a configuration averaging procedure. If the scatterers are sparsely distributed, (i.e., the concentration is small) we may employ a single scattering or first Born approximation.

Most of the previous computational results using scattering theory are, however,

often confined to the Rayleigh or low frequency limit, a specific angle of incidence, single scattering approximation and simple geometries such as circular cylinders and spheres. Our methodology which employs the T-matrix approach and suitable statistical averaging procedure, is completely general and can be used for multiple scattering analyses of acoustic, electromagnetic and elastic waves emphasizing the inherent unity of the approach. The analysis provides a powerful computational scheme for determining the coherent field characteristics of a medium containing arbitrary shaped scatterers over a wide range of frequencies.

We consider N number of discrete scatterers of arbitrary shape either randomly or periodically distributed. Both parallel and random orientations of the scatterers are considered. When N is very large (such that the number of scatterers per unit volume is finite), the T-matrix formalism is combined with a statistical analysis to provide a self-consistent multiple scattering theory. For a random distribution of scatterers we use a configurational averaging procedure, while for a periodic distribution we use a suitable lattice sum based on crystallographic theory. Inhomogeneous media in which scatterers have a specific orientation exhibit anisotropy and polarization effects. These effects are, however, nullified for the case of randomly oriented scatterers. The information about the orientation of the scatterers has been incorporated (in terms of a rotation matrix consisting of Euler angles) into the T-matrix of the scatterer itself thus making formalism a convenient computational scheme to study the anisotropic effects in an inhomogeneous medium. The statistically averaged equations are then solved using Lax's quasicrystalline approximation to yield the propagation characteristics of the coherent waves in the medium. Analytical results are obtained for the dispersion relation in the Rayleigh or low-frequency limit for both 2- and 3-dimensional scatterer geometries. Numerical results are presented for propagation speeds, attenuation and frequency dependent properties at higher frequencies to demonstrate the broad applicability of the T-matrix approach. The approach presented in this article has been successfully employed to many practical scattering problems on acoustic, electromagnetic and elastic wave fields (22-29).

II. T-MATRIX FORMULATION OF MULTIPLE SCATTERING

We consider N number of arbitrary shaped scatterers with a smooth surface S which are referred to a coordinate system as shown in Fig. 1. The scatterers may either be randomly distributed or periodically arranged. The orientation of the scatterers may be quite general. Here, we consider both specific and random orientations of the scatterers. In Fig. 1, O_i and O_j refer to the centers of the i-th and j-th scatterers, respectively and they are referred to the origin O by the spherical polar coordinates (r_i, θ_i, ϕ_i). For two dimensional problems, we use cylindrical polar coordinates (r_i, θ_i). $\bar{r}_{ij}$ refers to the vector connecting O_i and O_j. P is any point in the medium outside the scatterers (which we call the matrix medium).

We now describe the medium and the scatterers for all three wave fields. For an acoustic problem, we consider fluid scatterers immersed in another fluid, bubbles in a fluid, elastic or viscoelastic scatterers immersed in a fluid, etc. For an electromagnetic scattering problem, we consider dielectric scatterers in free space, dielectric scatterers embedded in a different dielectric medium, etc. For an elastic wave scattering problem, we consider elastic or viscoelastic inclusions embedded in another elastic or viscoelastic material, stress free or fluid filled cavities and cracks in an elastic or viscoelastic material, etc. The properties of the medium and the scatterers are given in terms of Lamé constants λ, μ and density ρ for an elastic material, compressibility λ_f and density ρ_f for non-viscous fluids

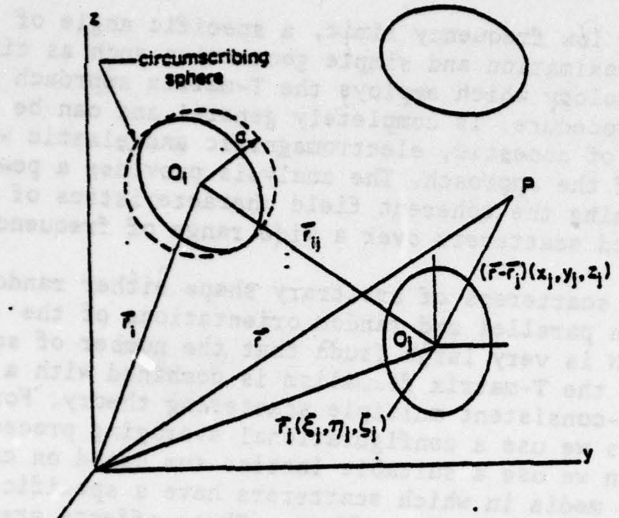


Fig. 1. Randomly distributed and oriented scatterers

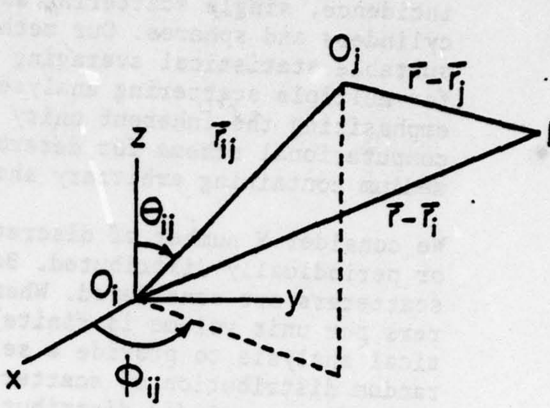


Fig. 2. Translation of the coordinate system of the j-th and i-th scatterers.

and relative dielectric constant ε_r and permeability μ_r with respect to free space describing dielectric medium. We use subscript 1 to denote these qualities inside the scatterers.

A time harmonic plane wave of unit amplitude and frequency ω is incident on the medium such that the direction of propagation of the incident waves is along the z-axis. The incident wave field may then be represented by

$$\vec{u}^{o} = e^{i(k_p z - \omega t)}\hat{z} + e^{i(k_s z - \omega t)}\hat{x} \tag{2.1}$$

where k_p and k_s are the compressional (longitudinal) and transverse (shear) wave numbers, respectively, and t is the time. The acoustic waves are purely compressional type and thus, the second term of Eq. (2.1) is set equal to zero; for electromagnetic waves which are transverse type, the first term on the right hand side of (2.1) is zero; for elastic waves which contain both compressional and transverse types, all the term of (2.1) are present. For acoustic and elastic wave problems, $\vec{u}^{o}$ refers to the incident displacement field vector, while for electromagnetic case, it refers to the incident electric field vector. In Eq. (2.1), we use the superscript (o) to indicate an incident wave. The wave numbers k_p and k_s are given by

$$k_p = \omega/c_p \quad ; \quad k_s = \omega/c_s \, , \tag{2.2}$$

respectively, where

$$c_p = \sqrt{(\lambda + 2\mu)/\rho} \qquad \text{(elastic waves)} \tag{2.3}$$

$$= \sqrt{\lambda_f/\rho_f} \qquad \text{(acoustic waves in fluids)} \tag{2.4}$$

$$c_s = \sqrt{\mu/\rho} \qquad \text{(elastic waves)} \qquad (2.5)$$

$$= C/\sqrt{\mu_r \varepsilon_r} \qquad \text{(electromagnetic waves)} \qquad (2.6)$$

In Eqs. (2.3) - (2.6), c_p and c_s refer to compressional and transverse wave velocities, respectively. In Eq.(6), C refers to the velocity of light in free space. The corresponding quantities inside the scatterers are differentiated by subscript 1. For brevity, we use the notation k_τ for the wave numbers ; $\tau = 1$ corresponds to compressional wave while $\tau = 2,3$ corresponds to shear wave.

When the wave impinges on a scatterer, part of the incident wave is scattered back into the matrix (medium outside the scatterer) and the rest is refracted into the scatterers and transmitted back into the matrix.We represent the scattered field by $\vec{u}^s(\vec{r})$ and the total field inside the scatterers by $\vec{u}^t(\vec{r})$. Since the incident wave has time dependence given by exp(-iωt), all these field quantities have the same time factor which will be suppressed henceforth. Both these fields and the incident wave field satisfy the vector Helmholtz equation, see for example (30). Even though for acoustic wave propagation in fluids one could work with velocity potential and scalar Helmholtz equation, we prefer to use the vector displacement vector field and the vector Helmholtz equation for the reasons mentioned in our previous papers (20,31,32). The problem at hand reduces to computing the total wave field at any point in the matrix satisfying the appropriate boundary condition on the surface of the scatterers and radiation conditions at infinity.

The total field at any point in the matrix (outside the scatterers) is the sum of the incident field and the fields scattered by all the scatterers. This is written as

$$\vec{u}(\vec{r}) = \vec{u}^o(\vec{r}) + \sum_{i=1}^{N} \vec{u}_i^s(\vec{r} - \vec{r}_i) \qquad (2.7)$$

where $\vec{u}_i^s(\vec{r} - \vec{r}_i)$ is the field scattered by the i-th scatterer to the point of observation $\vec{r}$. The field that excites the i-th scatterer is the incident field $\vec{u}^o$ plus the fields scattered from all other scatterers. The exciting-field term $\vec{u}^e$ is used to distinguish between the field actually incident on a scatterer and the external incident field $\vec{u}^o$ produced by a source at infinity. Thus, at a point $\vec{r}$ in the vicinity of the i-th scatterer, we write

$$\vec{u}_i^e(\vec{r}) = \vec{u}^o(\vec{r}) + \sum_{j\neq i}^{N} \vec{u}_j^s(\vec{r} - \vec{r}_j) \ , \quad a \leq |\vec{r} - \vec{r}_i| \leq 2a \qquad (2.8)$$

where 'a' is the radius of the imaginary sphere circumscribing a scatterer. In this analysis, we have assumed that there is no interpenetration of the imaginary spheres of radius 'a' which circumscribe each scatterer.

The T-matrix formalism of scattering we employ here is based on the extended boundary condition method which has been discussed by many authors in this book (33). This formalism differs from the eigen function expansion technique in that the same basis sets namely, the vector spherical wave functions for 3-D problems and the vector cylindrical wave functions for 2-D problems, may be used for scatterers of any closed boundary S with a continuous turning normal. The vector fields are expanded in terms of a complete set of basis functions which form

solutions to the vector Helmholtz equation and are given by

$$\vec{\psi}_{1\sigma mn}(\vec{r}) = \left(k_p/k_s\right)^{1/2} \xi_{mn} \nabla \left[h_n(k_p r)\, P_n^m(\cos\theta) \begin{pmatrix} \cos m\phi\,;\sigma = 1 \\ \sin m\phi\,;\sigma = 2 \end{pmatrix} \right] \tag{2.9}$$

$$\vec{\psi}_{2\sigma mn}(\vec{r}) = k_s \eta_{mn} \nabla \times \left[\vec{r}\, h_n(k_s r)\, P_n^m(\cos\theta) \begin{pmatrix} \cos m\phi\,;\sigma = 1 \\ \sin m\phi\,;\sigma = 2 \end{pmatrix} \right] \tag{2.10}$$

$$\vec{\psi}_{3\sigma mn}(\vec{r}) = (1/k_s)\, \nabla \times \vec{\psi}_{2\sigma mn}(\vec{r}) \tag{2.11}$$

where

$$\xi_{mn} = \left[\varepsilon_n \frac{(2n+1)\,(n-m)!}{4\pi(n+m)!} \right]^{1/2}$$

$$\eta_{mn} = \left[\frac{\varepsilon_n (2n+1)\,(n-m)!}{4\pi\, n(n+1)\,(n+m)!} \right]^{1/2} \tag{2.12}$$

with $\varepsilon_n = 1$ for $n = 0$ and $\varepsilon_n = 2$ for $n > 0$.

For brevity, we abbreviate these vector basis functions as $\vec{\psi}_{1\sigma mn}$; $\vec{\psi}_{2\sigma mn}$; $\vec{\psi}_{3\sigma mn} = \vec{\psi}_{\tau n}$; $\tau = 1,2,3$. For acoustic wave problems, we have only $\vec{\psi}_{1n}$; for electromagnetic waves, we have $\vec{\psi}_{2n}$ and $\vec{\psi}_{3n}$; for elastic waves, we have all three of them. In Eqs. (2.9 - 2.11), we have used spherical polar coordinates r,θ,ϕ with the origin of the coordinate system inside S, $h_n(1)$ are the spherical Hankel functions of order n, P_n^m are the associated Legendre polynomials and m is an integer that takes values 0, 1, 2, ..., n ; n = 0, 1, 2, ...,∞ for $\sigma = 1$ and n = 1,2,3,...,∞ for $\sigma = 2$. Field quantities that are regular at the origin are expanded in terms of the regular (Re) basis set (Re $\vec{\psi}_{\tau mn}$) obtained by replacing h_n in the above equations by j_n, the spherical Bessel functions of the first kind of order n. For two dimensional problems, we employ cylindrical basis functions, see, for example (31).

We now expand the exciting and scattered fields in terms of basis functions Re $\vec{\psi}_{\tau n}$ and $\vec{\psi}_{\tau n}$, respectively :

$$u_i^e(\vec{r}) = \sum_\tau \sum_{n=0}^{\infty} \sum_{\ell=-n}^{n} \sum_{\sigma=1}^{2} b^i_{\tau n\ell\sigma}\ \mathrm{Re}\,\vec{\psi}^i_{\tau n\ell\sigma}\,(\vec{r} - \vec{r}_i)$$

$$= \sum_{\tau n} b_{\tau n}^{\ell(i)}\ \mathrm{Re}\,\vec{\psi}^i_{\tau n\ell} \equiv \sum_{\tau n} b^i_{\tau n}\ \mathrm{Re}\,\vec{\psi}^i_{\tau n} \tag{2.13}$$

$$\vec{u}_j^s(\vec{r}) = \sum_{\tau n\ell} B_{\tau n}^{\ell(j)}\ \vec{\psi}^j_{\tau n\ell} \equiv \sum_{\tau n} B^j_{\tau n}\ \vec{\psi}^j_{\tau n} \tag{2.14}$$

where the superscripts i or j on the basis functions refer to expansions with respect to O_i and O_j, respectively, and $B^j_{\tau n}$ and $b^i_{\tau n}$ are unknown coefficients to be evaluated. The choice of basis set in (2.14) satisfies the radiation condition at infinity for the scattered field, while the choice in (2.13) satisfies the

regularity of the exciting field in the region $0<|\vec{r}-\vec{r}_i|<2a$.

Substituting (2.13) and (2.14) in (2.8), we obtain

$$\sum_{\tau n} b^i_{\tau n} \; \mathrm{Re}\, \vec{\psi}^i_{\tau n} = \vec{u}^o(\vec{r}-\vec{r}_i) + \sum_{j\neq i}^{N} \sum_{\tau n} B^j_{\tau n} \vec{\psi}^j_{\tau n} \tag{2.15}$$

It can be seen that the second series on the right-hand side of (2.15) are expressed with respect to the center of the j-th scatterer. In order to express them with respect to the i-th center, we employ the addition theorems of vector spherical harmonics for 3-D problems and of Bessel functions for 2-D problems. This translation has been discussed in detail in (34,35) and employed for acoustic, elctromagnetic and elastic wave problems (22-29). For the sake of uniformity, we reproduce the essential equations which translate the basis function from j-th center to the i-th center :

$$\begin{aligned}
\vec{\psi}^j_{1n} &= \vec{\psi}^j_{1n\ell} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A^{n\ell}_{\mu\nu} \; \mathrm{Re}\, \vec{\psi}^i_{1\nu\mu} \\
\vec{\psi}^j_{2n} &= \vec{\psi}^j_{2n\ell} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[B^{n\ell}_{\mu\nu} \; \mathrm{Re}\, \vec{\psi}^i_{2\mu\nu} + C^{n\ell}_{\mu\nu} \; \mathrm{Re}\, \vec{\psi}^i_{3\mu\nu} \right] \\
\vec{\psi}^j_{3n} &= \vec{\psi}^j_{3n\ell} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[C^{n\ell}_{\mu\nu} \; \mathrm{Re}\, \vec{\psi}^i_{2\mu\nu} + B^{n\ell}_{\mu\nu} \; \mathrm{Re}\, \vec{\psi}^i_{3\mu\nu} \right]
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
A^{n\ell}_{\mu\nu} &= \sum_q (-1)^{\mu} i^{\nu+q-\ell} (2\nu+1) a(n,\ell|-\mu,\nu|q) h_q(k_p r_{ij}) P^{n-\mu}_q (\cos\theta_{ij}) e^{i(n-\mu)\phi_{ij}} \\
B^{n\ell}_{\mu\nu} &= \sum_q (-1)^{\mu} i^{\nu+q-\ell} a(\ell,\nu,q) a(n,\ell|-\mu,\nu|q) h_q(k_s r_{ij}) P^{n-\mu}_q (\cos\theta_{ij}) e^{i(n-\mu)\phi_{ij}} \\
C^{n\ell}_{\mu\nu} &= \sum_q (-1)^{\mu+1} i^{\nu+q-\ell} b(\ell,\nu,q) a(n,\ell|-\mu,\nu|q,q-1) h_q(k_s r_{ij}) P^{n-\mu}_q (\cos\theta_{ij}) \\
&\quad\; e^{i(n-\mu)\phi_{ij}}
\end{aligned} \tag{2.17}$$

Expressions for $a(\ell,\nu,q)$, $b(\ell,\nu,q)$, $a(n,\ell|-\mu,\nu|q)$ and $a(n,\ell|-\mu,\nu|q,q-1)$ are given by Cruzan (34) and Fig. 2 depicts the geometry of the translation between the j-th and i-th scatterers. In Eq. (2.16), for acoustic wave scattering, we keep only $\vec{\psi}^j_{1n}$; for electromagnetic wave scattering, we keep $\vec{\psi}^j_{2n}$ and $\vec{\psi}^j_{3n}$; for elastic wave

scattering, we need all three of them. For two dimensional problems, the translation is given by

$$\psi^{j}_{\tau n} = \psi_{\tau n}(\vec{r}-\vec{r}_j) = (-1)^n \sum_{m} (-1)^m \, \mathrm{Re}\,\psi_{\tau m}(\vec{r}-\vec{r}_i)\, \psi_{\tau, n-m}(\vec{r}_i-\vec{r}_j) \qquad (2.18)$$

which contain cylindrical Hankel and Bessel functions. For details, we refer to (22, 26, 36). For the sake of simplicity we use an abbreviation $\sigma_{\tau n\, \tau' n'}$ to represent the translation of the basis functions from the j-th center to the i-th center and then we write

$$\vec{\psi}^{j}_{\tau n} = \vec{\psi}_{\tau n}(\vec{r}-\vec{r}_j) = \sum_{\tau' n'} \sigma_{\tau n\, \tau' n'}(\vec{r}_i-\vec{r}_j)\, \mathrm{Re}\,\vec{\psi}^{i}_{\tau' n'} \qquad (2.20)$$

It then remains to expand the incident wave also in the form of a series centered at the i-th scatterer. Referring to Fig. 1, we can rewrite Eq.(2.1) with $\exp(-i\omega t)$ term suppressed

$$\vec{u}^{0} = e^{ik_p(\zeta_i + z_i)}\hat{z} + e^{ik_s(\zeta_i + z_i)}\hat{x}\,. \qquad (2.20)$$

Expanding the terms $\exp(ik_p z_i)$ and $\exp(ik_s z_i)$ in (2.20) in terms of spherical Bessel functions and Legendre polynomials, and using the integral representation for spherical Bessel function of the first kind and orthogonality relations, we can express (2.20) in the following form, see for example Stratton (37) :

$$\vec{u}^{0} = \frac{e^{i k_p \zeta_i}}{i k_p} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} (2s+1)\, i^s \, \mathrm{Re}\,\vec{\psi}^{i}_{1ts}\, \delta_{t,0}$$

$$+ \frac{1}{2i}\, e^{i k_p \zeta_i} \sum_{s=1}^{\infty} \sum_{t=-s}^{s} \frac{2s+1}{s(s+1)}\, i^s \Big\{ \mathrm{Re}\,\vec{\psi}^{i}_{2ts}\left[\delta_{t,1} + s(s+1)\delta_{t,-1}\right]$$

$$+ \frac{1}{k_s}\, \mathrm{Re}\,\vec{\psi}^{i}_{3ts}\left[\delta_{t,1} - s(s+1)\,\delta_{t,-1}\right]\Big\} \qquad (2.21)$$

where δ_{mn} is the Kronecker δ. For two dimensional problems, the incident waves are expanded by using Fourier series expansion in complex form, see for example (22,26). For the sake of simplicity, we write the incident wave field in terms of expansion co-efficients $a_{\tau n}$ as follows

$$\vec{u}^{0} = \sum_{\tau n} a_{\tau n}\, \mathrm{Re}\,\vec{\psi}^{i}_{\tau n}\, e^{i k_\tau \cdot \vec{r}_i} \qquad (2.22)$$

In writing (2.22), we have used the concept that for $\vec{k} = k\hat{z}$, $k\hat{z}\cdot\vec{r} = k\hat{z}\cdot(\vec{r} - \vec{r}_i) + k\hat{z}\cdot\vec{r}_i$ and $\hat{z}\cdot\vec{r}_i = \zeta_i$. For acoustic wave scattering ($\tau = 1$), $a_{\tau n}$ is equal to the first term of Eq. (2.21) multiplying $\exp(i\, k_p\, \zeta_i)$ and Re $\vec{\psi}_{1ts}$; for electromagnetic waves ($\tau = 2,3$), $a_{\tau n}$ is equal to the second term of Eq. (2.21) multiplying $\exp(i\, k_s \zeta_i)$ and Re $\vec{\psi}_{\tau ts}$. Since the incident wave field is a known field, $a_{\tau n}$ are, hence, known expansion field coefficients. Substituting Eqs. (2.19) and (2.21) in (2.15) and using orthogonality of basis functions, we obtain the following relation between the unknown expansion coefficients $b_{\tau n}$ of the exciting field and the unknown expansion coefficients $B_{\tau n}$ of the scattered field :

$$b^i_{\tau n} = a_{\tau n}\, e^{i\vec{k}_\tau\cdot\vec{r}_i} + \sum_{j\neq i}^{N} \sum_{\tau' n'} B^j_{\tau' n'}\ \sigma_{\tau' n';\, \tau n}\, (\vec{r}_i - \vec{r}_j) \tag{2.23}$$

It has been shown by the previous papers on T-matrix approach that if the total field outside a scatterer is the sum of the incident and the scattered fields, the unknown scattererd field expansion coefficients can be related to the incident field expansion coefficients through the transition or T-matrix. We extend this definition to our present problem. Since $(\vec{u}^e_i + \vec{u}^s_i)$ is the total field at any point in the matrix, the expansion coefficients of the field scattererd by the i-th scatterer may be formally related to the coefficients of the field exciting the i-th scatterer through the T-matrix :

$$B^i_{\tau n} = \sum_{\tau' n'} T^i_{\tau n, \tau' n'}\ b^i_{\tau' n'} \tag{2.24}$$

In its expanded version, Eq. (2.24) can be written as

$$\begin{bmatrix} B^{\ell(i)}_{1n} \\ B^{\ell(i)}_{2n} \\ B^{\ell(i)}_{3n} \end{bmatrix} = \begin{bmatrix} (T^{11})^{\ell p(i)}_{nm} & (T^{12})^{\ell p(i)}_{nm} & (T^{13})^{\ell p(i)}_{nm} \\ (T^{21})^{\ell p(i)}_{nm} & (T^{22})^{\ell p(i)}_{nm} & (T^{23})^{\ell p(i)}_{nm} \\ (T^{31})^{\ell p(i)}_{nm} & (T^{32})^{\ell p(i)}_{nm} & (T^{33})^{\ell p(i)}_{nm} \end{bmatrix} \begin{bmatrix} b^{p(i)}_{1m} \\ b^{p(i)}_{2m} \\ b^{p(i)}_{3m} \end{bmatrix} \tag{2.25}$$

The elements of the T-matrix, $T_{\tau n, \tau' n'}$, involve surface integrals which can be evaluated in closed form for cylindrical geometry (2-D) and spherical geometry (3-D), while for obstacles of arbitrary shape they can be evaluated numerically. The T-matrix for a single scatterer is of the form

$$T = -Q(\mathrm{Re},\, \mathrm{Re})\ Q^{-1}(\mathrm{Ou},\, \mathrm{Re}) \tag{2.26}$$

where Q(Re, Re) and Q(Ou, Re) are matrices which are functions of the surface S

of the scatterer and of the nature of the boundary conditions. This form is quite common for acoustic, electromagnetic and elastic wave problems, except when there are solid-fluid interfaces. A detailed derivation of the T-matrix for a solid in a fluid and fluid in a solid can be found in the paper by Varadan (31).

With the scattered field coefficients $B^j_{\tau n}$ expressed in terms of exciting field coefficients $b^j_{\tau n}$ and the T-matrix as given by (2.24), Eq. (2.23) gives the exciting field formulation of the multiple scattering. If we multiply both sides of Eq. (2.23) by the T-matrix, then we obtain the scattered field formulation of multiple scattering which may be written as

$$B^i_{\tau n} \equiv B^{\ell(i)}_{\tau n} = \sum_{\tau'' n''} T^i_{\tau n, \tau'' n''} \Big[a_{\tau'' n''} \exp(i\, \vec{k}_\tau \cdot \vec{r}\,) + \sum_{j \neq i}^{N} \sum_{\tau' n'} B^j_{\tau' n'} \, \sigma_{\tau' n', \tau'' n''} (\vec{r}_i - \vec{r}_j) \Big] . \tag{2.27}$$

In its expanded version, Eq. (2.27) can be written as

$$\begin{bmatrix} B^{\ell(i)}_{1n} \\ B^{\ell(i)}_{2n} \\ B^{\ell(i)}_{3n} \end{bmatrix} = \begin{bmatrix} (T^{11})^{\ell p(i)}_{nm} & (T^{12})^{\ell p(i)}_{nm} & (T^{13})^{\ell p(i)}_{nm} \\ (T^{21})^{\ell p(i)}_{nm} & (T^{22})^{\ell p(i)}_{nm} & (T^{23})^{\ell p(i)}_{nm} \\ (T^{31})^{\ell p(i)}_{nm} & (T^{32})^{\ell p(i)}_{nm} & (T^{33})^{\ell p(i)}_{nm} \end{bmatrix} \begin{bmatrix} \phi^i_{1mp} \\ \phi^i_{2mp} \\ \phi^i_{3mp} \end{bmatrix} \tag{2.28}$$

where

$$\phi^i_{1pm} = (2m+1)\, i^m \frac{e^{i k_p \zeta_i}}{i k_p} \delta_{p,0} + \sum_{j \neq 1}^{N} \sum_{\nu=0}^{\infty} \sum_{\mu-\nu}^{\nu} B^{\mu}_{1\nu} A^{\nu\mu}_{mp}$$

$$\phi^i_{2mp} = \frac{2m+1}{m(m+1)}\, i^m \frac{e^{i k_s \zeta_i}}{2i} \left[\delta_{p,1} + m(m+1)\, \delta_{p,-1} \right] + \sum_{j=1}^{N}{}' \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[B^{\mu}_{2\nu} B^{\nu\mu}_{mp} + B^{\mu}_{3\nu} C^{\nu\mu}_{mp} \right] \tag{2.29}$$

$$\phi^i_{3mp} = \frac{2m+1}{m(m+1)}\, i^m \frac{e^{i k_s \zeta_i}}{2i} \left[\delta_{p,1} - m(m+1)\, \delta_{p,-1} \right] + \sum_{j=1}^{N}{}' \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[B^{\mu}_{2\nu} C^{\nu\mu}_{mp} + B^{\mu}_{3\nu} B^{\nu\mu}_{mp} \right]$$

In Eq. (2.29), $\sum'$ denotes the sum over all scatterers except the i-th. Thus, we have eliminated the unknown exciting field expansion coefficients through the use of the T-matrix resulting in a set of equations involving the expansion coefficients of the scattered field only. If the scatterers are all identical, then $T^i = T^j = T$. If they are of different sizes, we introduce a suitable size distribution for the scatterers and find average value of the T-matrix.

It can be seen from Eq. (2.27) that the scattererd field coefficients explicitly depend on the position and orientation of the scatterers. Depending on what information we choose to put into the distribution function, we can make our model more realistic but the analysis in turn gets more complicated. Here, we consider both random and periodic distributions with specific and random orientations.

III. RANDOM DISTRIBUTION OF SCATTERERS WITH SPECIFIC ORIENTATION

We consider N number of randomly distributed homogeneous scatterers. The scatterers are assumed to have a specific orientation (with major axis parallel to the y-axis) as shown in Fig. 3.

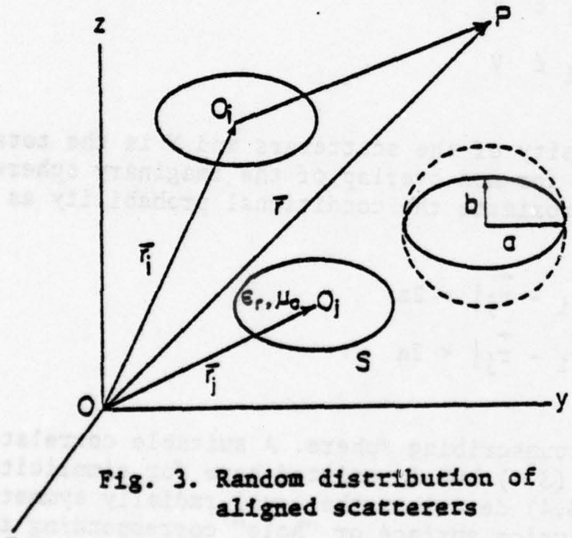


Fig. 3. Random distribution of aligned scatterers

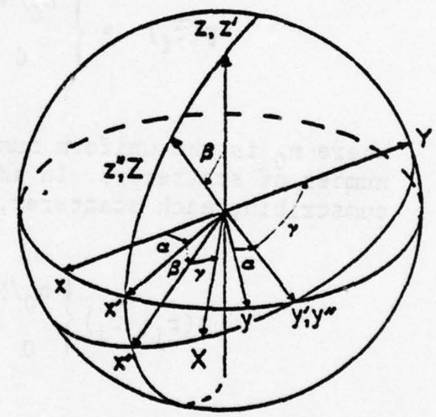


Fig. 4. Euler angles

Distribution functions are given by the total probability density of finding the scatterers at positions $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ etc., which in turn can be written in terms of conditional probabilities :

$$p(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N) = p(\bar{r}_i)\, p(\bar{r}_1\ \bar{r}_2, \ldots'. \ldots, \bar{r}_N|\bar{r}_i)$$

$$= p(\bar{r}_i)\ p(\bar{r}_j|\bar{r}_i)\, p(\bar{r}_1, \bar{r}_2, \ldots', \ldots', \bar{r}_N|\bar{r}_j, \bar{r}_i) \qquad (3.1)$$

In Eq. (3.1), $p(\vec{r}_i)$ denote the probability of finding a scatterer $\vec{r}_i$ and $p(\vec{r}_j|\vec{r}_i)$ denotes the conditional probability of locating a scatterer at $\vec{r}_j$ given that a scatterer is located at $\vec{r}$ etc. A prime in the first of Eq. (24) means $\vec{r}_i$ is absent while two primes in the second of Eq. (24) means both $\vec{r}_i$ and $\vec{r}_j$ are absent. We also denote the configurational averaging of a statistical quantity f as

$$\langle f\rangle_i = \int_V \ldots \int_V f\; p(\vec{r}_1, \vec{r}_2, \ldots', \vec{r}_N \,|\, \vec{r}_i)\, d\vec{r}_1\, d\vec{r}_2 \ldots', \ldots, d\vec{r}_N$$

$$\langle f\rangle_{ij} = \int_V \ldots \int_V f\; p(\vec{r}_1, \vec{r}_2, \ldots', \ldots', \ldots \vec{r}_N \,|\, \vec{r}_i, \vec{r}_j)\, d\vec{r}_1 d\vec{r}_2 \ldots', \ldots', \ldots d\vec{r}_N \qquad (3.2)$$

Equation (3.2a) implies that we have averaged over all scatterers except the i-th, while Eq. (3.2b) implies that we average over all scatterers except the i-th and j-th, and so on.

If the scatterers are randomly distributed, the positions of all scatterers are equally probable within volume V, and hence its distribution is uniform with probability density

$$p(\vec{r}_i) = \begin{cases} n_0/N, & \vec{r}_i \in V \\ 0, & \vec{r}_i \notin V \end{cases} \qquad (3.3)$$

where n_0 is the uniform number density of the scatterers and N is the total number of scatterers. In addition, for non-overlap of the imaginary spheres circumscribing each scatterer, we approximate the conditional probability as follows :

$$p(\vec{r}_j|\vec{r}_i) \begin{cases} n_0/N, & |\vec{r}_i - \vec{r}_j| > 2a \\ 0, & |\vec{r}_i - \vec{r}_j| < 2a \end{cases} \qquad (3.4)$$

where 'a' is the radius of the circumscribing sphere. A suitable correlation in the position may also be added to (3.4) but is omitted here for simplicity. The form of pair correlation in Eq. (3.4) describes the usual radially symmetric distribution function with an exclusion surface or "hole" corresponding to a sphere of radius 2a. Other types of exclusion surfaces are discussed in recent papers by Twersky (8, 9). We refer the reader to Twersky's comments on choosing radially symmetric statistics for non-symmetrical scatterers, namely making the region of non-overlap of two scatterers to be a circle or sphere circumscribing the scatterers.

Multiplying both sides of Eq. (2.26) by the probability density given by Eq. (3.1) and using (3.2) - (3.4), we obtain the configurational average of the scattered field coefficients :

$$\langle B^i_{\tau n}\rangle_i = \sum_{\tau'' n''} T_{\tau n, \tau'' n''} \Big[a_{\tau'' n''}\, e^{i\, \vec{k}_\tau . \vec{r}}$$

$$+ \frac{1}{V} \sum_{j \neq i}^{N} \sum_{\tau' n'} \int_{V'} \langle B^{j}_{\tau' n'} \rangle_{ij} \; \sigma_{\tau' n', \tau'' n''} \; d\vec{r}_j \Big] \tag{3.5}$$

where V′ denotes the volume of the medium excluding the volume of a sphere of radius 2a. For identical scatterers, $\sum_{j \neq i}^{N}$ can be replaced by (N-1). The above equation indicates that the configurational average with one scatterer fixed is given in terms of the configurational average with two scatterers fixed. We choose to solve these equations by using Lax's quasicrystalline approximation where we assume that there is no correlation between the scatterers except that of non-overlapping which can be written mathematically as

$$\langle B_{\tau n} \rangle_{ij} = \langle B_{\tau n} \rangle_{j} \tag{3.6}$$

Equation (3.6) also implies that the neighbourhood of every scatterer is the same. For identical scatterers, Eq. (3.5) may be rewritten using Eq. (3.6) as

$$\begin{aligned} \langle B^{i}_{\tau n} \rangle_i = \sum_{\tau'' n''} T_{\tau n, \tau'' n''} \Big[& a_{\tau'' n''} \; e^{i \vec{k}_{\tau} \cdot \vec{r}_i} \\ & + \frac{N-1}{V} \sum_{\tau' n'} \int_{V'} \langle B^{j}_{\tau n} \rangle_j \; \sigma_{\tau' n', \tau'' n''} \; d\vec{r}_j \Big] \end{aligned} \tag{3.7}$$

In its expanded version, Eq. (3.7) can be written as

$$\begin{bmatrix} \langle B^{\ell(i)}_{1n} \rangle_i \\ \langle B^{\ell(i)}_{2n} \rangle_i \\ \langle B^{\ell(i)}_{3n} \rangle_i \end{bmatrix} = \begin{bmatrix} (T^{11})^{\ell p}_{nm} & (T^{12})^{\ell p}_{nm} & (T^{13})^{\ell p}_{nm} \\ (T^{21})^{\ell p}_{nm} & (T^{22})^{\ell p}_{nm} & (T^{23})^{\ell p}_{nm} \\ (T^{31})^{\ell p}_{nm} & (T^{32})^{\ell p}_{nm} & (T^{33})^{\ell p}_{nm} \end{bmatrix} \begin{bmatrix} \langle \Phi^{i}_{1mp} \rangle \\ \langle \Phi^{i}_{2mp} \rangle \\ \langle \Phi^{i}_{3mp} \rangle \end{bmatrix} \tag{3.8}$$

where

$$\begin{aligned} \langle \Phi^{i}_{1mp} \rangle = {} & (2m+1)\, i^{m} \frac{e^{i k_p \zeta_i}}{i k_p} \delta_{p,0} \\ & + \frac{1}{V} \sum_{j=1}^{N'} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \int_{V'} \langle B^{\mu(j)}_{1\nu} \rangle_{ij} \; A^{\nu\mu}_{mp} \; d\vec{r}_j \\ & \ldots \end{aligned} \tag{3.9}$$

This is a system of integral equations for the unknown coefficients $\langle B^{i}_{\tau n} \rangle_i$. Similar expressions for the average exciting field coefficients may be obtained from the exciting field formalism of multiple scattering.

IV. PROPAGATION CHARACTERISTICS OF THE AVERAGE WAVES IN THE MEDIUM

To solve the integral equations given by (3.7), we consider the inhomogeneous medium with discrete scatterers as a homogeneous continuum and assume that the average coherent wave is a plane wave propagating with an effective wave number K in the same direction as the incident plane wave. We can thus write

$$< B^{i}_{\tau n} > = X_{\tau n} \, e^{i \vec{K} \cdot \vec{r}_i} \tag{4.1}$$

where $X_{\tau n}$ is the amplitude of the coherent wave.

Substituting Eq. (4.1) in (3.7), employing divergence theorem to convert the volume integral in (3.7) to surface integrals and using the extinction theorem which cancels the incident wave, see for example (8,9,22), we obtain a set of simultaneous coupled homogeneous equations for the coefficients $X_{\tau n}$ given by

$$X_{\tau n} = c \sum_{\tau'' n''} \sum_{\tau' n'} \sum_{q=|n'-n''|}^{|n'+n''|} X_{\tau' n'} \, T_{\tau n, \tau'' n''} \, C^{q}_{\tau' n', \tau'' n''} \frac{JHq}{(k^2_{\tau'} - K^2) a^2} \tag{4.2}$$

where $c = 4\pi a^3 n_0/3$ is the effective spherical concentration of the scatterers per unit volume, C^q is an expression containing Wigner coefficients, and

$$JH_q = 2 k_{\tau'} a \; j(2 Ka) h'_q(2 k_{\tau'} a) - 2 K a \; h_q(2 k_{\tau'} a) j'_q (2 Ka) \tag{4.3}$$

Equation (4.2) can be written in its expanded form as follows :

$$X^{\ell}_{1n} = 6c \sum_{q=|n_1-m|}^{|n_1+m|} \sum_{m=0}^{\infty} \sum_{p=-m}^{m} \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} (-1)^p (-i)^q$$

$$i^{m_1+m+q-n_1} \delta_{m_1 p} \; JH_q \left\{ \frac{1}{(k^2_p - K^2)a^2} \quad X^{m_1}_{1n_1} (T^{11})^{\ell p}_{nm} \right.$$

$$a(m_1, n_1 \, | -p, m \, | \, q) + \frac{1}{(k^2_s - K^2)a^2} \left\{ X^{m_1}_{2n_1} \left[(T^{12})^{\ell p}_{nm} \; a(n_1, m, q) \right. \right.$$

$$a(m_1, n_1 \, | -p, m \, | \, q) - (T^{13})^{\ell p}_{nm} \; b(n_1, m, q)$$

$$\left. a(m_1, n_1 \, | -p, m \, | \, q, q-1) \right] + X^{m_1}_{3n_1} \left[(T^{13})^{\ell p}_{nm} \; a(n_1, m, q) \right.$$

$$a(m_1,n_1|-p,m|q) - (T^{12})^{\ell p}_{nm} \quad b(n_1,m,q)$$

$$a(m_1,n_1|-p,m|q,\ q-1) \Big]\Big\}\Big\} \tag{4.4a}$$

$$X^{\ell}_{2n} = \ldots \tag{4.4b}$$

$$X^{\ell}_{3n} = \ldots \tag{4.4c}$$

Equation (4.4b) can be obtained from (4.4a) by replacing T^{11}, T^{12}, T^{13} by T^{21}, T^{22}, T^{23}, while (4.4c) can be obtained by replacing T^{11}, T^{12}, T^{13} by T^{31}, T^{32}, and T^{33}. For acoustic wave problem, we get uncoupled equation for X^{ℓ}_{1n} ; for electromagnetic wave problem, we get coupled equations in terms of X^{ℓ}_{2n} and X^{ℓ}_{3n}; for elastic wave problem, we obtain coupled equations in terms of X^{ℓ}_{1n}, X^{ℓ}_{2n} and X^{ℓ}_{3n}. Similar equations can also be obtained from the average exciting field coefficients, if one chooses to use exciting field formalism.

Equation (4.2) is a system of simultaneous linear homogeneous equations for the unknown amplitudes $X_{\tau n}$. For a nontrivial solution, we require that the determinant of the truncated coefficient matrix vanishes, which yields an equation for the effective wave number K in terms of k_τ and the T-matrix of the scatterer. This is the dispersion relation for the scatterer filled medium. Equation (32) is a general expression valied for any arbitrary shaped scatterer, since the T-matrix is the only factor that contains information about the exact shape and boundary conditions at the scatterer. Thus the formalism presented here is valid for all the three wave fields. The effective wave number K obtained in the analysis is a complex quantity, the real part of which relates to the phase velocity, while the imaginary part relates to attenuation of coherent waves in the medium.

V. PERIODIC DISTRIBUTION WITH PARALLEL ORIENTATION

For periodic distribution the analysis will introduce the geometry of the lattice leading to different results for different packing, and is similar to the one employed in crystallographic theories, see for example (38,39). Here we apply lattice sum and include contributions from nearest neighbors. One couls use other refinements following the theories presented in (8,9,38,39). The averaging over the position of individual scatterers can be easily accomplished since there is no restriction on the position of one scatterer.

A procedure for evaluating a lattice sum over a simple lattice is to obtain by direct summation the contributions by the lattice points within a certain radius R_1 given by

$$\frac{4}{3} \; \frac{\pi R_1{}^3}{V} \doteq N_0 \tag{5.1}$$

and to replace the summation over the points beyond this radius by an integral. In Eq. (5.1), N_0 is the number of scatterers counting the point $\vec{r}_j = 0$ and V is the

volume of the unit cell, (see (40) for a given type of an array). The analysis is presented in (41,42) and the resulting scattered field coefficients are given by

$$B^i_{\tau n} = \sum_{\tau'' n''} T^i_{\tau n, \tau'' n''} \Big[a_{\tau'' n''} \, e^{i \vec{k}_\tau \cdot \vec{r}_i} + \sum_{j \neq i}^{N_0}{}' \sum_{\tau' n'} B^j_{\tau' n'} \, \sigma_{\tau' n', \tau'' n''} (\vec{r}_i - \vec{r}_j) + \frac{1}{V} \int_{R_1}^{\infty} \sum_{\tau' n'} B^j_{\tau' n'} \, \sigma_{\tau' n', \tau'' n''} (\vec{r}_i - \vec{r}_j) \, d\vec{r}_j \Big] \quad (5.2)$$

To obtain a solution to the above equation, we assume a plane wave propagating with an effective wave number K in the same direction as the incident wave :

$$B^i_{\tau n} = X_{\tau n} \, e^{i \vec{K} \cdot \vec{r}_i} \quad (5.3)$$

Substituting Eq. (5.3) in Eq. (5.2) and following the procedure we had outlined in the previous section, we obtain a set of simultaneous coupled homogeneous equations for the coefficients $X_{\tau n}$ given by

$$X_{\tau n} = c \sum_{\tau'' n''} \sum_{\tau' n'} \sum_{q=|n'-n''|}^{|n'+n''|} X_{\tau' n'} \, T_{\tau n, \tau'' n''} \, C^q_{\tau' n', \tau'' n''} \frac{1}{(k_\tau^2 - K^2) a^2} \Big[R_q + \frac{(k_\tau^2 - K^2) a^2}{c} P_q \Big] \quad (5.4)$$

where

$$R_q = k_\tau R_1 \, j_q(K R_1) \, h_q(k_\tau R_1) - K R_1 \, h_n(k_\tau R_1) \, j_n'(K R_1)$$

$$P_q = \sum_j e^{i \vec{K} \cdot \vec{r}_j} \, \sigma_{\tau' n', \tau'' n''} \quad (5.5)$$

and c is the concentration as defined before.

VI. ARBITRARY ANGLE OF INCIDENCE AND RANDOM ORIENTATION

Inhomogeneous media in which scatterers have a specific orientation will exhibit anisotropy and polarization effects when the waves are incident at arbitrary angles to the symmetry axes of the scatterers. When the incident wave direction is along the symmetry axis of the oblate spheroidal scatterers, we found that the effect on polarization is zero, and the coherent wave propagates with an effective wave

number K as if it propagates in an effective homogeneous and isotropic continuum, see for example (25). When the scatterers are randomly oriented, the effects of anisotropy and polarization are also nullified (29).

For symmetrically oriented scatterers, we define the Euler's angles α,β,γ of the symmetry axes of the scatterers (x,y,z) with respect to the laboratory co-ordinate system (X,Y,Z), see Fig.4. All quantities that are referred to the x,y,z system are distinguished by a $\wedge$. Then, one could write the spherical harmonics using the relation operator as follows (for details, we refer to (43)) :

$$Y_{\ell m\sigma} = \sum_{m'\sigma'} D_{mm'\sigma\sigma'}(\alpha,\beta,\gamma)\ \hat{Y}_{\ell m'\sigma'} \tag{6.1}$$

where D is the rotation matrix associated with rotation operator. The rotation operator leaves the length of the position vector $|\vec{r}|$ invariant. The rotation matrix can be easily incorporated into the T-matrix to obtain a new T-matrix which can be written as

$$T = (D^t)^{-1}\ \hat{T}\ (D^t) \tag{6.2}$$

where D^t is the matrix transpose of D. Equation (6.2) gives the desired relation between the T-matrices evaluated with respect to the two set of coordinate axes. $\hat{T}$ is independent of position and orientation and is hence the same for identical scatterers. The matrices T, however, *is different* if the orientation of the scatterers is not the same. Substituting Eq. (6.2) in Eq. (4.2) and Eq. (5.4), we obtain the dispersion relation, phase velocity, coherent wave attenuation and polarization effects for an arbitrary angle of incidence for random and periodic distribution, respectively. This is one of the basic advantage of formulating the multiple scattering in terms of a T-matrix.

For random orientation, one has to average over all Euler angles α,β,γ, and the information can thus be incorporated into the T-matrix :

$$\langle T\rangle = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} D(\alpha,\beta,\gamma)\ \hat{T}\ D^{-1}(\alpha,\beta,\gamma)\ \sin\beta\ d\beta\ d\gamma\ d\alpha\ . \tag{6.3}$$

With Eq. (6.3), Eqs. (4.2) and (5.4) yield the dispersion relation for both random and periodic distributions, respectively, with random orientation.

VII. RESULTS AND CONCLUSIONS

Using the theories outlines in previous sections, we present some analytical and numerical results for variety of 2-D and 3-D problems in all three wave fields to show the broad applicability of our multiple scattering approach. The sample examples given may find applications in many fields of engineering and science as outlined in the introduction. We present results for two dimensional parallel cylinders of elliptical cross section randomly and periodically (hexagonal array) distributed with specific and random orientation. The wave is incident normal to

the cylinders. The polarization of the wave is along the axis of the cylinder (SH-waves). We also consider spherical, oblate spheroidal scatterers of various aspect ratios randomly distributed with specific and random orientations. At low frequencies, analytical expressions are derived for the effective wave number in the average medium as a function of the geometry, the material properties and the angle of orientation of the scatterers. The formulation is ideally suited for numerical computation of phase velocity and attenuation at higher frequencies as evidenced by the results presented here and in (22 - 29). In addition, we present numerical results for dynamic shear moduli for 2-D case as a function of frequency using the T-matrix approach and the work of Bedeux and Mazur (44) and Varadan and Vezzetti (45).

VIIa. Rayleigh or Low Frequency Limit

In the Rayleigh or low frequency limit, the size of the scatterers is considered to be small when compared to the incident wavelength. It is then sufficient to take only the lowest order coefficient in the expansion of the fields. In this limit, the elements of the T-matrix can be obtained in closed form for various simple shapes (46). It can be shown that at low frequencies, only $X_{\tau 0}$, $X_{\tau 1}$ and $X_{\tau,-1}$ of Eq. (4.2) or Eq. (5.4) make a contribution. After some manipulations of the resulting 3 × 3 determinant, we obtain the dispersion relations :

(i) Parallel Elliptic Cylinders (Elastic SH-Waves)

a) Specific Orientation of Cross Section (see Fig. 5 and paper (23))

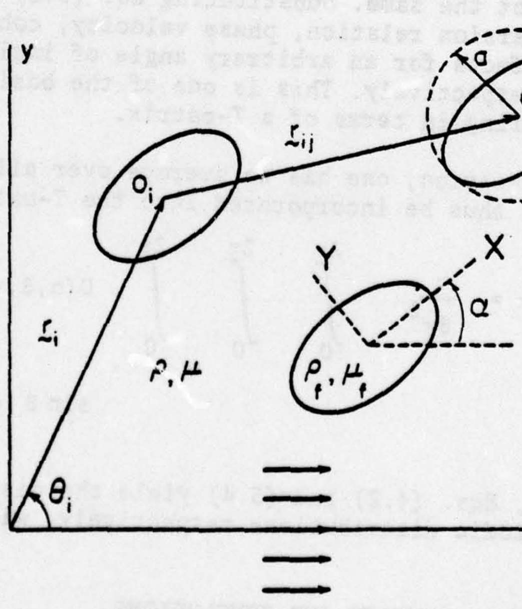


Fig. 5. Random distribution of aligned scatterers - oblique incidence

$$\frac{K^2}{k_s^2} = \frac{e_1 d_1 + e_2 d_2}{b_1 d_1 + b_2 d_2} \tag{7.1}$$

where

$$e_1 = (2ic/\pi^2)\, a_1\, e^{-i2\alpha} \left[1 - 2k_s^2 a^2\, \ln(k_s a)\right] \times$$
$$\times \left[\pi - 4ic\, S_0 + 8ic\, S_0\, k_s^2\, a^2\, \ln(k_s a)\right]$$

$$e_2 = (1/\pi^2) \left[\pi - 4ic\, S_0 + 8ic\, S_0\, k_s^2\, a^2\, \ln(k_s a)\right] \times$$
$$\times \left[\pi - 2ic \left\{1 - 2k_s^2 a^2\, \ln(k_s a)\right\} (S_1 + a_1\, e^{-i2\alpha})\right]$$

$$d_1 = (4c/\pi)\, a_1 \sin 2\alpha \left[1 - 2k_s^2 a^2\, \ln(k_s a)\right]$$

$$d_2 = (1/\pi) \left[\pi - 2ic \left\{1 - 2k_s^2 a^2\, \ln(k_s a)\right\} (S_1 - a_1\, e^{-i2\alpha})\right]$$

$$b_1 = (2ic/\pi^2)\, S_1 \left[\pi + 8ic\, S_0\, k_s^2 a^2\, \ln(k_s a)\right]$$

$$b_2 = (1/\pi^2) \left[\pi + 8ic\, S_0\, k_s^2\, a^2\, \ln(k_s a)\right] \left[\pi + 2ic\, (S_1 + a_1\, e^{i2\alpha})\right]$$

$$S_0 = i(\pi/8)\, (d - 1)\, (b/a) \left[2 - (d - 1)\, k_s^2 ab\, \ln(k_s a)\right] \tag{7.2}$$

$$S_1 = \frac{i\pi}{8} \frac{(1 - m)(1 + b/a)^2 (b/a)}{(b/a + m)(1 + mb/a)} \left\{1 + m - \left(\frac{1}{4}\right)(1 - m) \left[(1 + b^2/a^2) \times\right.\right.$$
$$\left.\left.\times\, (1 + m^2) + 4\, mb/a\right] k_s^2 ab\, \ln(k_s a)/(1 + mb/a)(b/a + m)\right\}$$

$$a_1 = \frac{i\pi}{8} \frac{(1 - m)(1 - b^2/a^2)(b/a)}{(b/a + m)(1 + mb/a)} \left[1 - m\, \frac{1}{4} \frac{(1 + b/a)^2 (1 - m^2)}{(b/a + m)(1 + mb/a)} \times\right.$$
$$\left. k_s^2 ab\, \ln(k_s a)\right]$$

and $c = \pi a^2 n_0$ is the concentration of the circumscribing circle.

b) Random Orientation

$$K^2/k_s^2 = (1 + c_1 t_1) \left[1 + c_1 (d - 1)\right] / (1 - c_1 t_1) \tag{7.3}$$

where

$$t_1 = (1 - m^2)(a + b)^2 / 4(b + ma)\, (mb + a) \tag{7.4}$$

(ii) Random Distribution of 2-D cracks (Elastic SH - Waves)

a) Parallel Orientation (7.5)

$$\frac{K^2}{k_s^2} = \frac{(4 + \pi a^2 n_0)^2 - (\pi a^2 n_0)^2\, e^{-i4\alpha} - 2i(\pi a^2 n_0)^2\, e^{-i2\alpha} \sin 2\alpha}{\left[4 - \pi a^2 n_0 (1 - e^{i2\alpha})\right] \left[4 + \pi a^2 n_0 (1 + e^{-i2\alpha})\right] - 2i(\pi a^2 n_0)^2 \sin 2\alpha}$$

b) Random Orientation

$$K^2/k_s^2 = (1 + \pi a^2 n_0/4)\ (1 - \pi a^2 n_0/4) \tag{7.6}$$

where 2a is the length of the crack and n_0 is its number density, and α is the angle the incident wave makes with the major axis. In these equations, d = ratio of densities of the scatterer to that of the matrix = ρ_1/ρ and m = ratio of shear moduli of the scatterer to that of the matrix = μ_1/μ and a and b are semi major and minor axes of the elliptic cylinder, respectively. The analysis for both compressional and shear waves incident normal to cylinders is presented in (27).

(iii) Spheres (elastic waves)

$$\left(\frac{K_p}{k_s}\right)^2 = \frac{(1 + 9c\,E_1)(1 + 3c\,E_0)\left[1 + 3c\,\frac{E_2}{2}\,(2 + \frac{3k_s^2}{k_p^2})\right]}{1 - 15c\,E_2\left[1 + 3c\,E_0\right] + \frac{3}{2}\,c\,E_2\,(2 + \frac{3k_s^2}{k_p^2})} \tag{7.7}$$

$$\left(\frac{K_s}{k_s}\right)^2 = \frac{(1 + 9c\,E_1)(1 + \frac{3}{2}\,c\,E_2\left[2 + \frac{3k_s^2}{k_p^2}\right])}{1 + \frac{3}{4}\,c\,E_2(4 - \frac{9\,k_s^2}{k_p^2})} \tag{7.8}$$

where

$$E_0 = \frac{1}{3}\ \frac{3\lambda + 2\mu - (3\lambda_1 + 2\mu_1)}{4\mu + 3\lambda_1 + 2\mu_1}$$

$$E_1 = \frac{1}{9}\left(\frac{\rho_1}{\rho} - 1\right)$$

$$E_2 = -\frac{\frac{4}{3}\,\mu\,(\mu_1-\mu)\quad 24\mu_1(\mu_1-\mu) - (\lambda_1+2\mu_1)\ (19\mu_1+16\mu)}{24\mu_1(\mu_1-\mu) - (\lambda_1+2\mu_1)\ (19\mu_1 + 16\mu)\cdot} \tag{7.9}$$

$$\times \frac{1}{4\mu(\mu_1-\mu) + 3(\lambda + 2\mu)\ (2\mu_1 + 3\mu)}$$

and c = 4 $\pi a^3 n_0/3$ is the concentration of spheres.

(iv) Spheres in Free Space (Electromagnetic Waves)

$$\left(\frac{K}{k_s}\right)^2 = \frac{1 + 2c\,\dfrac{\varepsilon_{r_1} - 1}{\varepsilon_{r_1} + 2}}{1 - c\,\dfrac{\varepsilon_{r_1} - 1}{\varepsilon_{r_1} + 2}} \tag{7.10}$$

(v) Spheroids in Free Space (Electromagnetic Waves)

Random Orientation

$$\left(\frac{K}{k_s}\right)^2 = \left[1 + \frac{\frac{3C}{2}\frac{\varepsilon_{r_1}-1}{\varepsilon_{r_1}+2}(f_2+f_3)}{1-\frac{3}{2}\frac{\varepsilon_{r_1}-1}{\varepsilon_{r_1}+2}f_1}\right] \cdot \left[1 - \frac{\frac{3c}{4}\frac{\varepsilon_{r_1}-1}{\varepsilon_{r_1}+2}(f_2+f_3)}{1-\frac{3}{2}\frac{\varepsilon_{r_1}-1}{\varepsilon_{r_1}+2}f_1}\right]^{-1} \tag{7.11}$$

where c is the concentration of the circumscribing spheres and f_1, f_2 and f_3 are functions of the eccentricity of the spheroids $e = [(a/b)^2 - 1]^{\frac{1}{2}}$ and are given in (25).

In the Rayleigh limit, the value of K as determined by the above dispersion relations is a real quantity for lossless material and a complex quantity for lossy material, and relates to phase velocity $V_p = \omega/K$. In this limit, we normally study the dependence of phase velocity on concentration, angle of incidence and aspect ratio of the scatterers. In Figs. 6 and 7, we have plotted the normalized phase velocity V_p/c_s versus concentration for various values of angle of incidence α for 2-D (SH-Wave) cylinders randomly distributed with parallel and random orientations. We assume the density ratio d = 2.53/2.72 and shear modulus ratio m = 25/3.87 (no loss in the material) for obtaining results in Fig. 6 while Fig. 7 gives the result for cracks. The general tendency of the phase velocity is to increase (for inclusions) and decrease (for cracks and cavities) as concentration increases. The results also indicate that phase velocity decreases as the angle of incidence α increases. For $\alpha = 0$ and $\alpha = \pi/2$, our results agree with well known results for cracks (3). It can easily be seen from Eq. (7.2) that for a given concentration and α, the phase velocity decreases with decreasing b/a. In Figs. 6 and 7, we have also plotted the corresponding results for random orientation.

Equation (7.10) is recognized as the dispersion relation of the Clausius-Mossotti form. The dispersion relation for spheroidal scatterers given by Eq. (7.11) appears to be new ; it reduces to Eq. (7.10) in the limit $e \to 0$. In Fig. 8, we have plotted the normalized phase velocity V_p/c_s for dielectric spheres and spheroidal scatterers in free space randomly distributed and oriented versus concentration. Both real and complex dielectric constants $(\varepsilon_r)_1$ of the scatterers are assumed. The values are taken for ice and water particles from (47).

The dispersion relation given in Eqs. (7.1) and (7.3) may also be useful in obtaining the effective shear modulus at low frequencies. Defining an average shear modulus $\langle\mu\rangle = \omega^2 \langle\rho\rangle K^2$ where $\langle\rho\rangle = c\,\rho_1 + (1 - c)\rho$ is the average density, we find the effective shear modulus of the composite medium containing cylindrical inclusions :

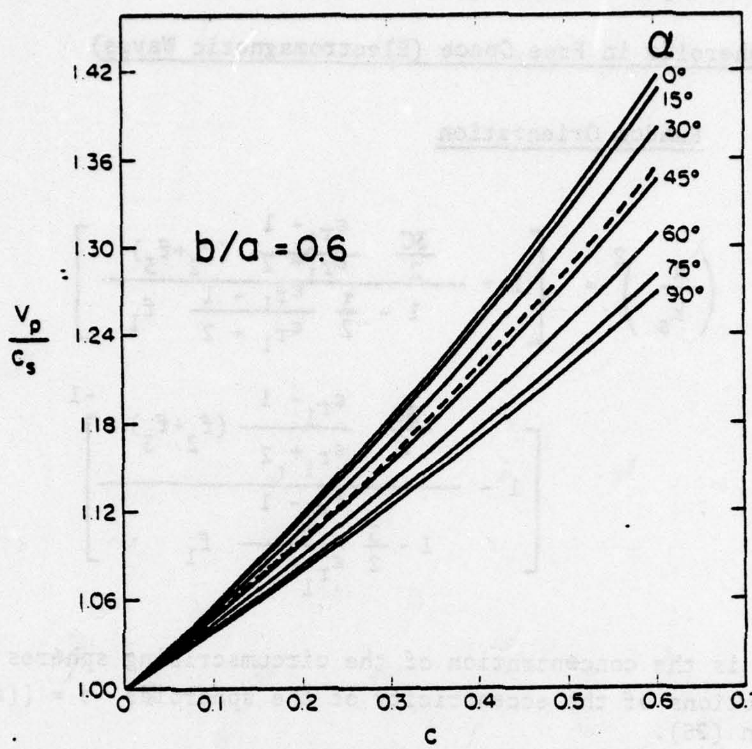


Fig. 6. Normalized phase velocity vs. concentration c for b/a = 0.6 and for various values of α; ——— aligned scatterers, ---randomly oriented scatterers

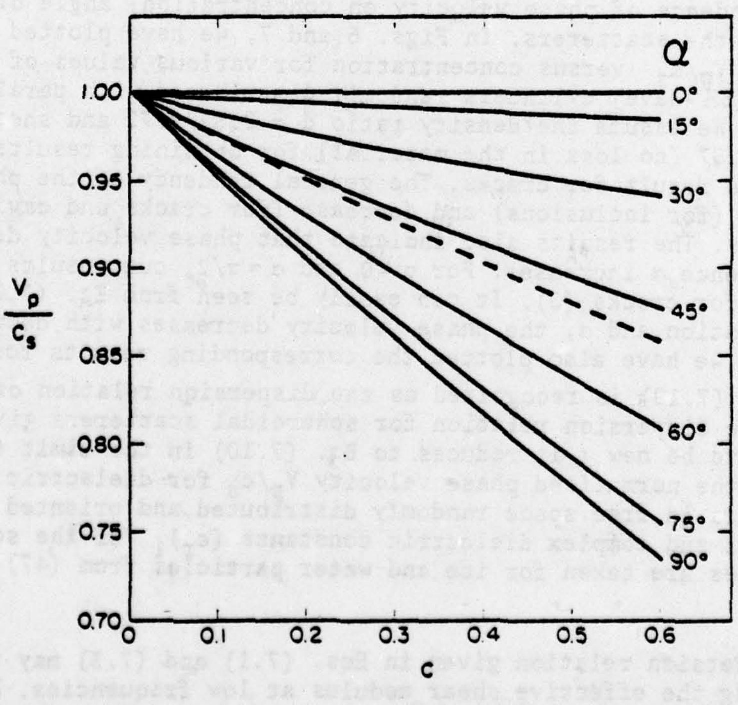


Fig. 7. Normalized phase velocity vs. concentration c for a random distribution of cracks; ——— aligned, --- random orientation

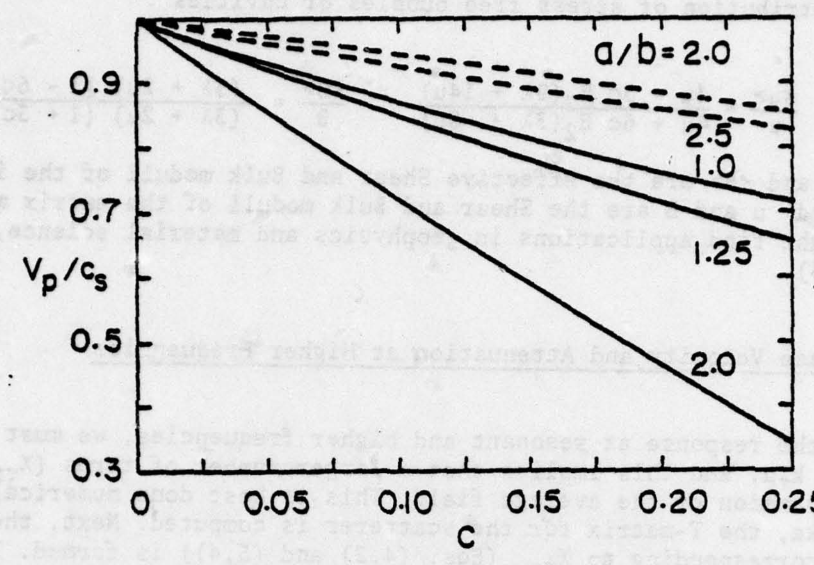


Fig. 8a. Normalized phase velocity vs. concentration for spherical and spheroidal dielectric scatterers in free space; —— lossy, --- lossless

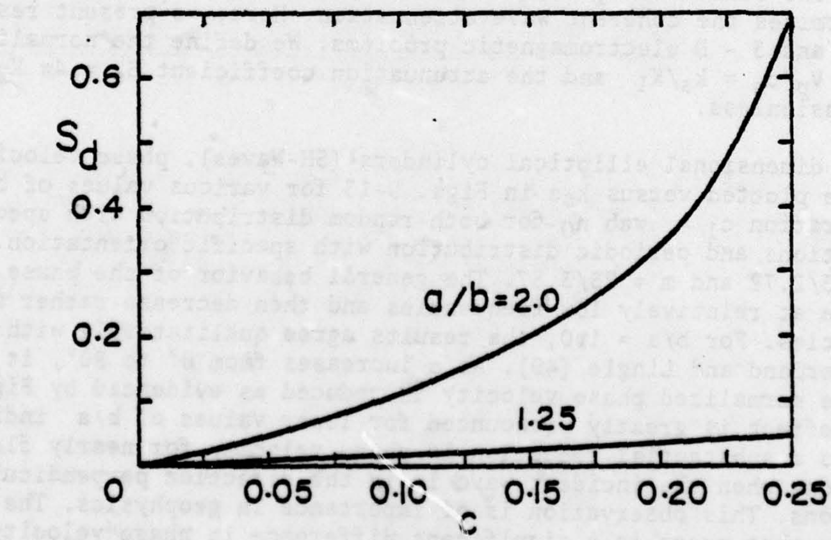


Fig. 8b. Attenuation S_d vs. concentration c for spheroidal lossy dielectric scatterers in free space

$$\langle\mu\rangle/\mu = (\langle\rho\rangle/\rho)\,(k_s/K)^2 \tag{7.12}$$

for more detail, we refer to (28).

Using the dispersion relation given in Eqs. (7.7) and (7.8), one could also

compute the effective Shear and Bulk moduli of an elastic material containing a random distribution of stress free bubbles or cavities :

$$\frac{\langle\mu\rangle}{\mu} = \frac{4\mu - 3c\,E_2(9\lambda + 14\mu)}{4\mu + 6c\,E_2(3\lambda + 8\mu)} \;;\; \frac{\langle B\rangle}{B} = \frac{(3\lambda + 2\mu)\,[1 - 6c\,E_0]}{(3\lambda + 2\mu)\,[1 + 3c\,E_0]} \qquad (7.13)$$

where $\langle\mu\rangle$ and $\langle B\rangle$ are the effective Shear and Bulk moduli of the inhomogeneous medium, and μ and B are the Shear and Bulk moduli of the matrix medium. This result might find applications in geophysics and material science, see for example Chaban (48).

VIIb. Phase Velocity and Attenuation at Higher Frequencies

To study the response at resonant and higher frequencies, we must consider higher powers of $k_\tau a$, and this implies that a larger number of terms ($X_{\tau n}$) must be kept in the expansion of the average field. This is best done numerically. For a given value of ka, the T-matrix for the scatterer is computed. Next, the coefficient matrix M corresponding to $X_{\tau n}$ (Eqs. (4.2) and (5.4)) is formed. The complex determinant of the coefficient matrix is computed using standard Gauss elimination techniques. For a given $k_\tau a$, the root of the equation det M = 0 is searched in the complex K plane ($K_1 + iK_2$) using Muller's method. Good initial guesses were provided by the Rayleigh limit solutions at low values of $k_\tau a$ and these could be used systematically to obtain convergence of roots at increasingly higher values of $k_\tau a$. The real part K_1 determines the phase velocity, while the imaginary part K_2 determines the coherent wave attenuation. Here, we present results for 2 - D SH-Wave and 3 - D electromagnetic problems. We define the normalized phase velocity as $V_p/c_s = k_s/K_1$ and the attenuation coefficient $S_d = 4\pi\,K_2/K_1$ so as to make it dimensionless.

For two dimensional elliptical cylinders (SH-Waves), phase velocity and attenuation are plotted versus $k_s a$ in Figs. 9-13 for various values of b/a, α and actual concentration $c_1 = \pi ab\,n_0$ for both random distribution with specific and random orientations and periodic distribution with specific orientation. We assume d = 2.53/2.72 and m = 25/3.87. The general behavior of the phase velocity is to increase at relatively low frequencies and then decrease rather sharply for higher frequencies. For b/a = 1.0, the results agree qualitatively with those obtained by Sutherland and Lingle (49). As α increases from 0° to 90°, it has been found that the normalized phase velocity is reduced as evidenced by Fig. 9. The anisotropic effect is greatly pronounced for lower values of b/a indicating that there is a substantial reduction in phase velocity for nearly flat oriented inclusions when the incident wave is in the direction perpendicular to the inclusions. This observation is of importance in geophysics. The results in Fig. 9 indicate that there is a significant difference in phase velocity for various orientations and that the corresponding values for random orientation lie in between 0° and 90° orientations.

In Figs. 10 and 12, we have plotted the coefficient of attenuation S_d versus $k_s a$. These results indicate that anisotropic effect is greatly pronounced ; substantial reduction of attenuation is observed when α = 90°. The general behavior of the damping or attenuation is to increase rapidly with frequency at low frequency range and decrease sharply or fluctuates slightly when frequency is increased. The results also indicate that there is a tendency for the attenuation to increase rapidly again at higher frequencies. The attenuation calculations for random orientation lie between 0° and 90° orientations, see Fig. 10. In Fig. 10, we have also plotted the coefficient of attenuation versus $k_s a$ for 2 - D cracks

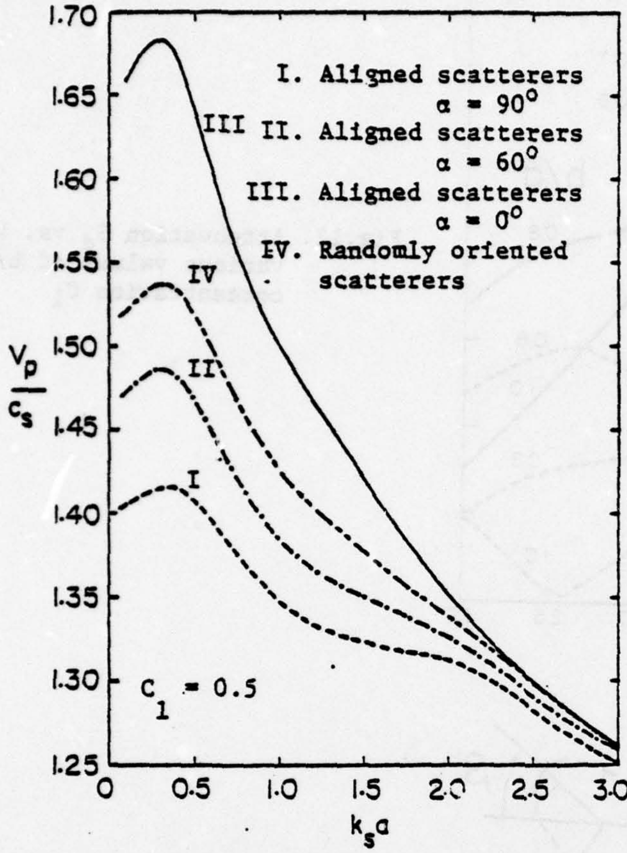


Fig. 9. Normalized phase velocity vs. $k_s a$ for a random distribution of aligned and randomly oriented scatterers

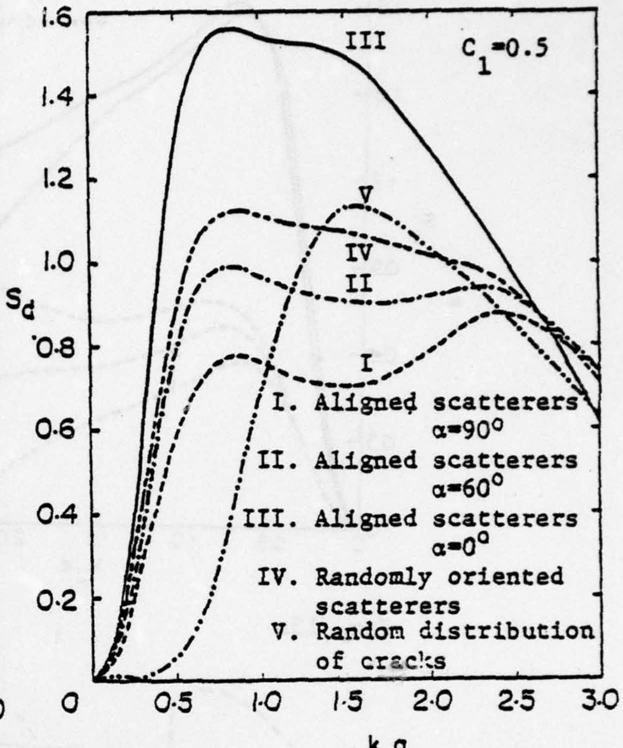


Fig. 10. Attenuation S_d vs. $k_s a$ for a random distribution of aligned and randomly oriented scatterers and cracks.

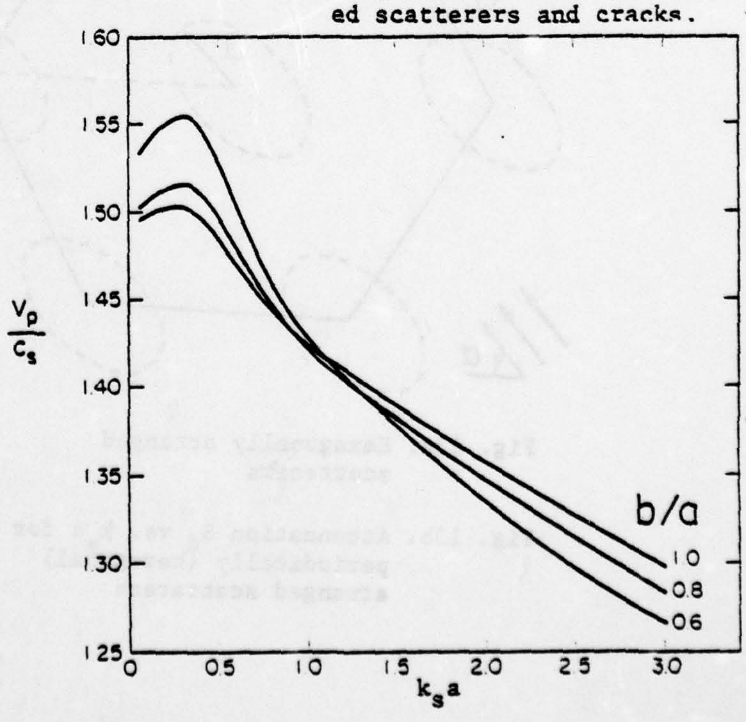


Fig. 11. Normalized phase velocity vs. $k_s a$ for α=45° and C_1=0.5

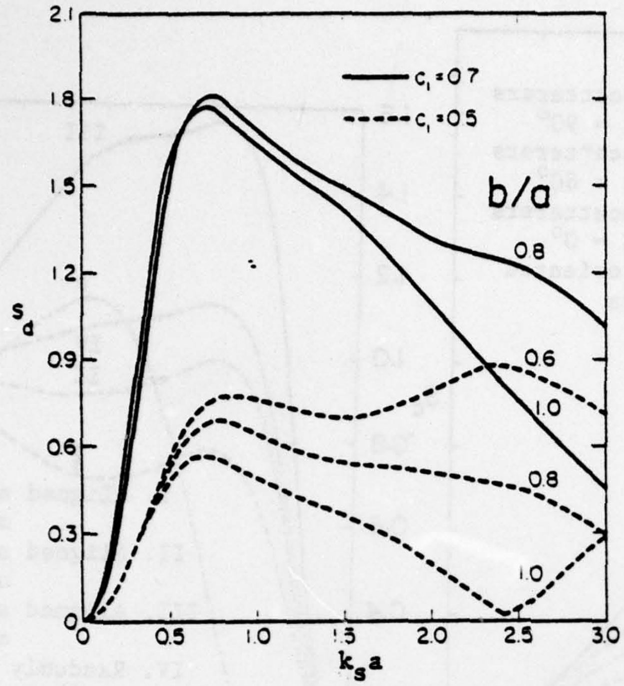


Fig. 13a

Fig.12. Attenuation S_d vs. $k_s a$ for various values of b/a and concentration C_1

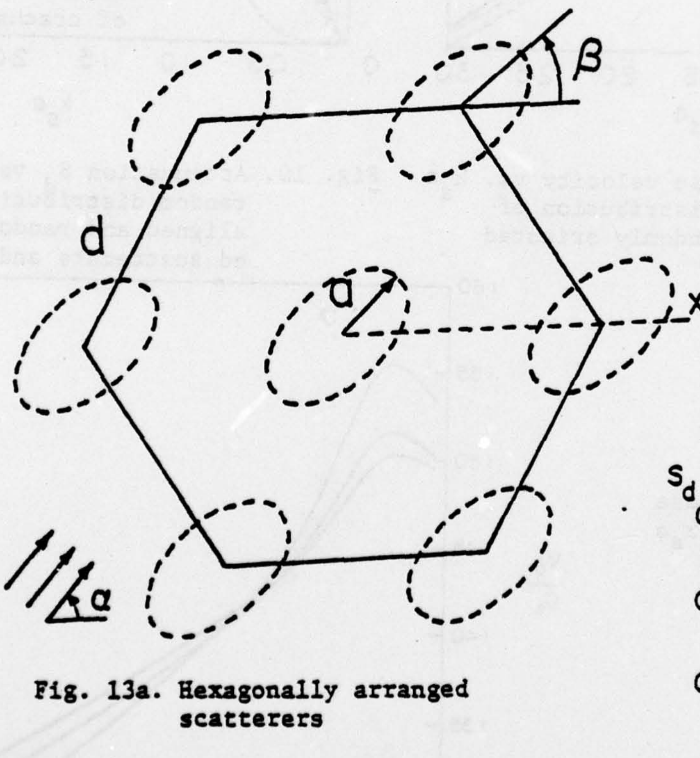


Fig. 13b

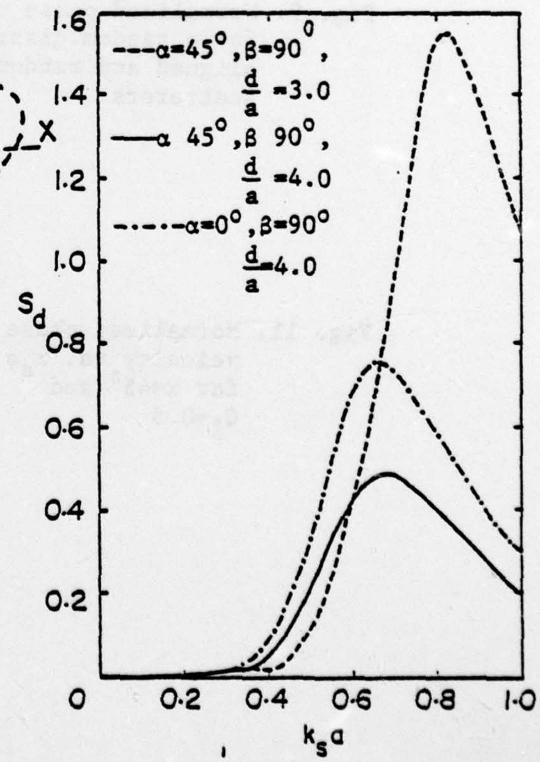


Fig. 13a. Hexagonally arranged scatterers

Fig. 13b. Attenuation S_d vs. $k_s a$ for periodically (hexagonal) arranged scatterers

randomly distributed.

The frequency at which S_d begins to decrease is usually referred to as cut off frequency of the first pass band. At the cut off frequency, S_d decreases because energy begins to pass into the second pass band, see the explanations in (49,50). For periodic distribution, these cut off frequencies seem to move towards lower $k_s a$ as the intercylindrical spacing between the cylinders increase, see Fig. 13.

In Figs. 14 - 17, we have plotted phase velocity and attenuation coefficients for spherical and spheroidal dielectric scatterers in free space randomly distributed with parallel and random orientations. The calculations were performed for both lossless and lossy dielectric scatterers. For lossless scatterers, we assume dielectric constant $\epsilon_r = 3.168$ which corresponds to ice at microwave frequencies. The imaginary part of the dielectric constant for ice is relatively small when compared to the real part, see Ray (47). For lossy case, we consider rain particles which have complex dielectric constants given as functions of temperature and frequency. The complex dielectric constants of rain particles are taken from the paper by Ray (47). For our calculations, we assume the temperature to be equal to 5°C.

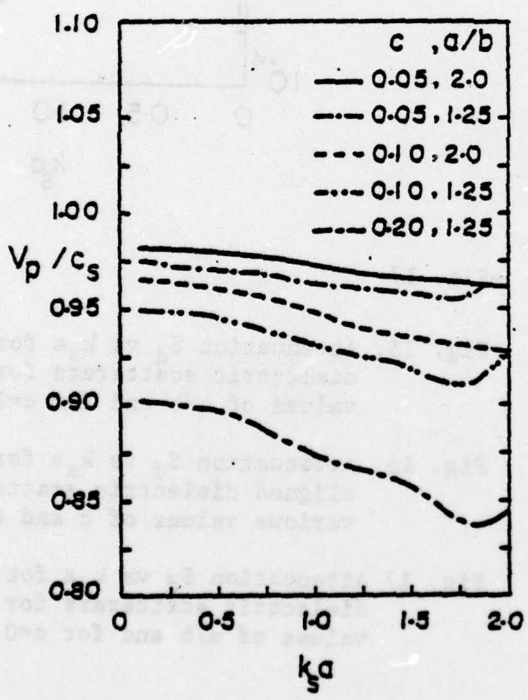


Fig. 14a. Normalized phase velocity vs $k_s a$ for aligned spheroidal dielectric scatterers

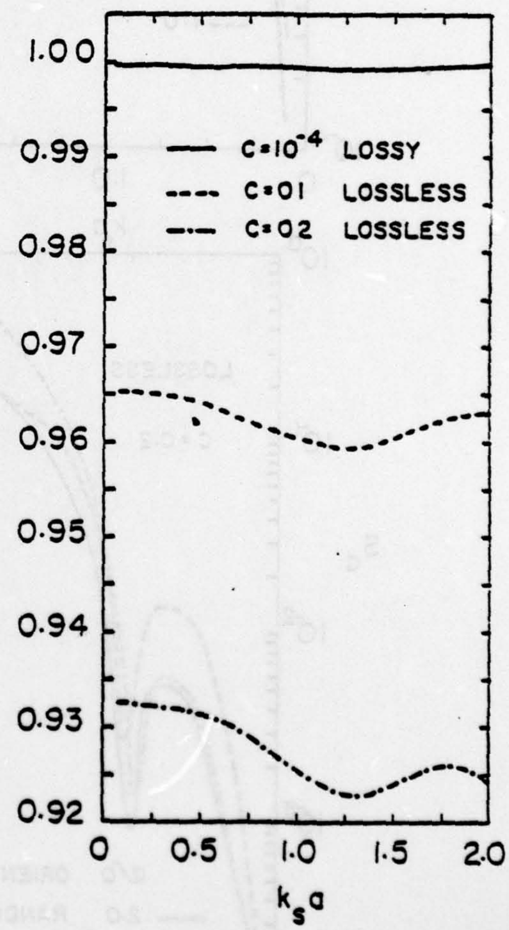
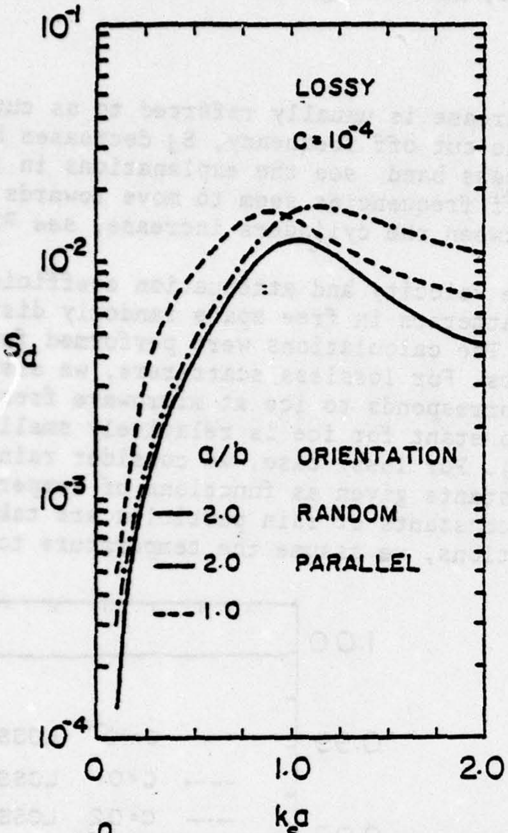


Fig. 14b. Normalized phase velocity vs $k_s a$ for randomly oriented spheroidal dielectric scatterers



←Fig. 15

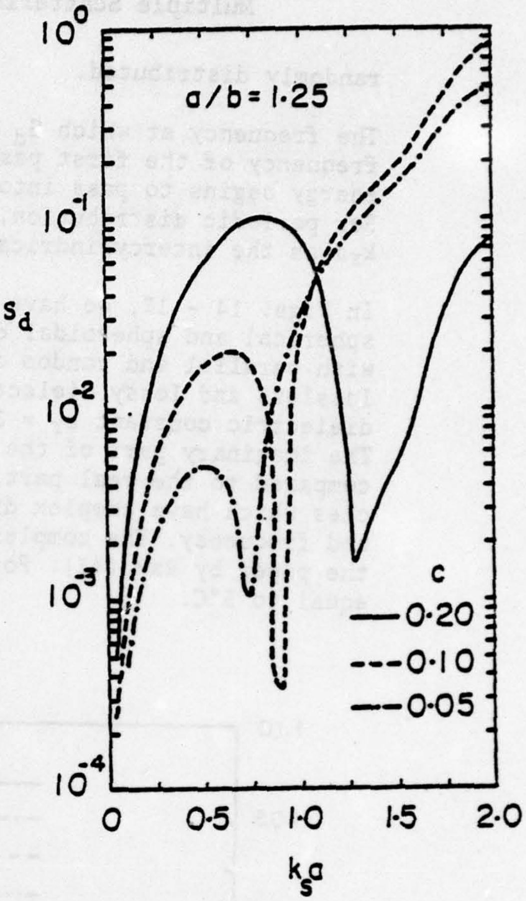
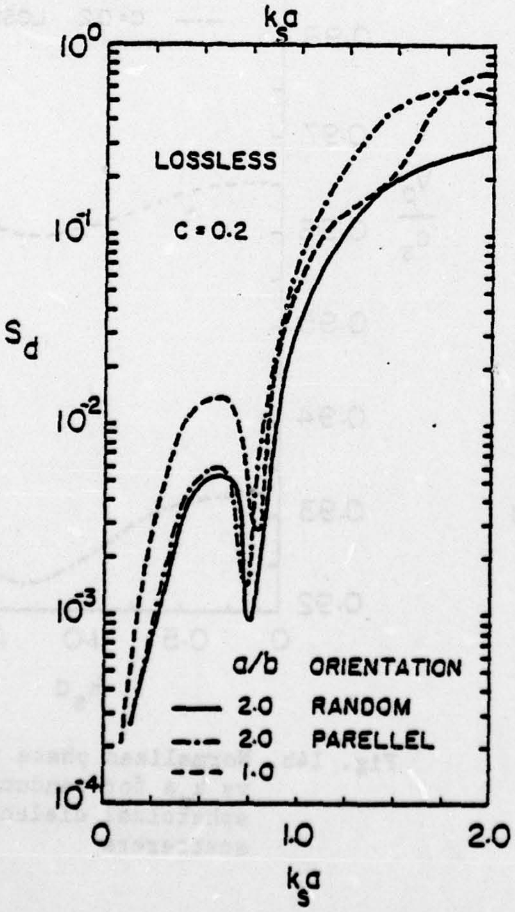


Fig. 16→



←Fig. 17

Fig. 15. Attenuation S_d vs $k_s a$ for lossy dielectric scatterers for various values of a/b and for $c=10^{-4}$

Fig. 16. Attenuation S_d vs $k_s a$ for lossless aligned dielectric scatterers for various values of c and a/b=1.25

Fig. 17 Attenuation S_d vs $k_s a$ for lossless dielectric scatterers for various values of a/b and for c=0.2

In Fig. 15, we show the attenuation coefficient S_d for lossy dielectric scatterers as a function of $k_s a$ for concentration $c = 10^{-4}$. The results are plotted for spherical and oblate spheroidal scatterers randomly distributed with both parallel and random orientations. For the case of parallel orientation, the spheroids are assumed to be oriented with their axes of revolution along the z-axis (see Fig. 3), and the incident wave is also assumed to propagate along the z-axis. In both specific and random orientation cases, it is well known that the bulk medium is not anisotropic and is characterized by a single wave number K. However, when the wave is incident obliquely to parallelly oriented scatterers, we obtain the anisotropic effects (just like the 2 - D case discussed above) and the bulk medium will then be characterized by an effective wave number with two different polarizations.

In Figs. 16 and 17, we have plotted S_d for lossless spheroidal dielectric scatterers for two different concentrations, $c = 0.1$ and 0.2, and for two different aspect ratios, $a/b = 1.25, 2$. The significant feature in the case of $c = 0.2$ is the presence of a sharp null at $k_s a = 0.75$.

The multiple scattering analysis presented so far can also be extended to compute effective dynamic moduli of a composite medium (28). Bedeaux and Mazur (44) have obtained expressions for the average dielectric tensor in the medium and Varatharajulu (Varadan) and Vezzetti (45) have shown that for a given statistical model the zeros of the propagator in this model medium yield the dielectric tensor defined by Bedeaux and Mazur (44). In (28), we have extended this discussion to elastic wave propagation wherein we have given sample calculations for dynamic shear moduli as a function of $k_s a$.

In conclusion, we have presented a multiple scattering formalism for coherent wave propagation of acoustic, electromagnetic and elastic waves through an inhomogeneous medium composed of either random or periodic distribution of 2-dimensional and 3-dimensional scatterers. The scatterers are assumed to have either specific or random orientation. An important advantage is realized through the use of the T-matrix to characterize the scattering properties of any one scatterer. The methodology adapted in this analysis is general and can be used to include such effects as scatterer size distribution, depolarization, etc.

ACKNOWLEDGEMENTS

This work was supported in part by NOAA under Grant No: 04-78-B01-21 and in part by the U.S. Office of Naval Research under contract No: N00014-78-C-0559. The author would like to acknowledge helpful discussions with Professor V.V. Varadan and Dr. V.N. Bringi.

REFERENCES

(1) A. Ishimaru (1978), Wave Propagation and Scattering in Random Media, 2, Academic Press, New York.

(2) Lord Rayleigh, On the transmission of light through an atmosphere containing small particles in suspension, and on the origin of the blue of the sky, Philos. Mag. 47, 375 (1899).

(3) J.P. Watt, G.F. Davies and R.J. O'Connell, The elastic properties of composite materials, Rev. Geophys. Space Phys. 14, 541 (1976).

(4) L.L. Foldy, The multiple scattering of waves I. General theory of isotropic scattering by randomly distributed scatterers, Phys. Rev. 67, 107 (1945).

(5) M. Lax, Multiple scattering of waves II. Effective field in dense systems, Phys. Rev. 85, 621 (1952).

(6) V. Twersky, On scattering and reflection of sound by rough surfaces, J. Accoust. Soc. Am. 29, 209 (1957).

(7) V. Twersky, On scattering of waves by random distributions. I. Free-space scatterer formalism, J. Math. Phys. 3, 700 (1962).

(8) V. Twersky, Coherent scalar field in pair-correlated random distribution of aligned scatterers, J. Math. Phys. 18, 2468 (1977).

(9) V. Twersky, Coherent electromagnetic waves in pair-correlated random distribution of aligned scatterers, J. Math. Phys. 19, 215 (1978).

(10) F.C. Karal, Jr. and J.B. Keller, Elastic, electromagnetic and other waves in a random medium, J. Math. Phys. 5, 537 (1964).

(11) S.K. Bose and A.K. Mal, Longitudinal shear waves in a fiber-reinforced composite, Int. J. Solids and Structures 9, 1075 (1973).

(12) S.K. Bose and A.K. Mal, Elastic waves in a fiber-reinforced composite, J. Mech. Phys. Solids. 22, 217 (1974).

(13) A.K. Mal and S.K. Bose, Dynamic elastic moduli of a suspension of imperfectly bonded spheres, Proc. Camb. Phil. Soc. 76, 587 (1974).

(14) S.K. Datta, Scattering of a random distribution of inclusions and effective elastic properties, *Continuum Models of Discrete Systems*, University of Waterloo Press (1977).

(15) D.J. Vezzetti and J.B. Keller, Refractive index, attenuation, dielectric constant, and permeability for waves in a polarizable medium, J. Math. Phys. 8, 1861 (1967).

(16) P.C. Waterman and R. Truell, Multiple scattering of waves, J. Math. Phys. 2, 512 (1961).

(17) J.G. Fikioris and P.C. Waterman, Multiple scattering of waves. II. Hole corrections in the scalar case, J. Math. Phys. 5, 1413 (1964).

(18) B. Peterson and S. Ström, T-matrix for electromagnetic scattering from an arbitrary number of scatterers and representations of E(3), Phys. Rev. D. 8, 3661 (1973).

(19) B. Peterson and S. Ström, Matrix formulation of acoustic scattering from an arbitrary number of scatterers, J. Accoust. Soc. Am. 56, 771 (1974).

(20) B. Peterson, V.V. Varadan and V.K. Varadan, On the multiple scattering of waves from obstacles with solid-fluid interfaces, Recent Developments in Classical Wave Scattering - Focus on the T-matrix Approach (Edited by V.K. Varadan and V.V. Varadan), Pergamon Press, New York (1979).

(21) J.D. Achenbach, Waves and vibrations in directionally reinforced composites, Composite Materials (Edited by L.J. Broutman and R.H. Crock), Academic Press, New York, 2 (1972).

(22) V.K. Varadan, V.V. Varadan and Y.H. Pao, Multiple scattering of elastic waves by cylinders of arbitrary cross section. I. SH-waves, J. Accoust. Soc. Am. 63, 1310 (1978).

(23) V.K. Varadan and V.V. Varadan, Frequency dependence of elastic (SH-) wave velocity and attenuation in anisotropic two phase media, Int. J. Wave Motion. 1, 53 (1979).

(24) V.K. Varadan, Scattering of elastic waves by randomly distributed and oriented scatterers, J. Accoust. Soc. Am. 65, 655 (1979).

(25) V.K. Varadan, V.N. Bringi and V.V. Varadan, Coherent electromagnetic wave propagation through randomly distributed dielectric scatterers, Phys. Rev. D. 19, 2480 (1979).

(26) V.K. Varadan and V.V. Varadan, Dynamic elastic properties of a medium containing a random distribution of obstacles : Scattering matrix theory, The Materials Science Center Report # 2740, Cornell University, New York (1976).

(27) V.K. Varadan and V.V. Varadan, Multiple scattering of elastic waves by cylinders of arbitrary cross section. II. P- and SV- waves, The Materials Science Center Report # 2937, Cornell University, New York (1977).

(28) V.K. Varadan and V.V. Varadan, Characterization of dynamic shear modulus in inhomogeneous media using ultrasonic waves, Proceedings of the First International Symposium on Ultrasonic Materials Characterization, National Bureau of Standards, Washington, D.C., in press.

(29) V.V. Varadan and V.K. Varadan, Multiple scattering of electromagnetic waves by randomly distributed and oriented dielectric scatterers, submitted for publication.

(30) V.V. Varadan and V.K. Varadan, Acoustic, electromagnetic and elastic field equations pertinent to scattering problems, in Recent Developments in Classical Wave Scattering - Focus on the T-matrix Approach (Edited by V.K. Varadan and V.V. Varadan), Pergamon Press, New York (1979).

(31) V.V. Varadan, Elastic wave scattering, in Recent Developments in Classical Wave Scattering - Focus on the T-matrix Approach (Edited by V.K. Varadan and V.V. Varadan), Pergamon Press, New York (1979).

(32) B. Peterson, V.V. Varadan and V.K. Varadan, Scattering of acoustic waves by elastic and viscoelastic obstacles immersed in a fluid, Int. J. Wave Motion, in press.

(33) V.K. Varadan and V.V. Varadan (1979), Recent Developments in Classical Wave Scattering - Focus on the T-matrix Approach, Pergamon Press, New York.

(34) O.R. Cruzan, Translational addition theorems for spherical vector wave equations, Q. Appl. Math. 20, 33 (1962).

(35) B. Peterson and S. Ström, T-matrix formulation of electromagnetic scattering from multilayered scatterers, Phys. Rev. D. 10, 2670 (1974).

(36) G.N. Watson (1945), A Treatise on the Theory of Bessel Functions, The Macmillan Company, New York.

(37) J.A. Stratton (1941), Electromagnetic Theory, McGraw-Hill, New York.

(38) M.J.P. Musgrave (1970), Crystal Acoustics - Introduction to the Study of Elastic Waves and Vibrations in Crystals, Holden-Day.

(39) A.K. Ghatak and L.S. Kothari (1972), An Introduction to Lattice Dynamics, Addition-Wesley Publishing Company, Massachusetts.

(40) C. Kittel (1956), Introduction to Solid State Physics, John Wiley and Sons, Inc.

(41) V.K. Varadan and V.V. Varadan, Self-consistent approach to elastic wave propagation : application to periodic composite media, submitted for publication.

(42) V.K. Varadan, Dispersion of longitudinal shear waves in a medium containing periodic arrays of cylinders of arbitrary cross section Proc. 14th. Annual Meeting of the Soc. Eng. Sci., (1977).

(43) A.R. Edmonds (1957), Angular Momentum in Quantum Mechanics, Princetown University Press, Princetown, New Jersey.

(44) D. Bedeaux and P. Mazur, On the critical behavior of the dielectric constant for a nonpolar fluid, Physica, 67, 23 (1973).

(45) V.V. Varatharajulu (Varadan) and D.J. Vezzetti, Approach of the statistical theory of light scattering to the phenomenological theory, J. Math. Phys. 17, 232 (1976).

(46) V.V. Varadan and V.K. Varadan, Low Frequency expressions for acoustic wave scattering using Waterman's T-matrix method, J. Acoust. Soc. Am. in press.

(47) S.P. Ray, Broad band complex refractive index of ice and water, Appl. Optics. 11, 1836 (1972).

(48) I.A. Chaban, Calculation of effective parameters of micro inhomogeneous media by the self-consistent field method, Soviet Phys. Acoust. 11, 81 (1965).

(49) H.J. Sutherland and R. Lingle, Geometric dispersion of acoustic waves by a fiberous composite, J. Comp. Mater. 6, 490 (1972).

(50) F.C. Moon and C.C. Mow, Wave propagation in a composite material containing dispersed rigid spherical inclusions, The Rand Corporation Report, RM-6139-PR, Santa Monica, California (1970).